Basic Graph Theory

Hrishik Koley, Bikram Halder

Contents

Chapter 0

Introduction

These notes provide a fundamental introduction to graph theory, serving as a prerequisite for the Winter Reading Project (WRP) on Random Graphs. While it offers a solid foundation, this is not a substitute for comprehensive graph theory books. The content focuses specifically on topics essential for the WRP.

For a deeper understanding, consider referring to *"Introduction to Graph Theory"* by *Douglas B. West*, *"Graphs and Matrices"* by *R.B. Bapat*, and *Benny Sudakov*'s notes available at this link.

chapter 9 contains simple graph theory questions to reinforce your understanding. Additionally, chapter 10 features a list of open problems for those seeking a challenge.

While some sections cover interesting concepts beyond the scope of the Random Graphs WRP, marked by $**$, you are free to explore them at your discretion.

Chapter 1

Basic Definitions

1.1 Notation and Preliminaries

Definition 1.1.1 ▶ Graph

A (simple) graph $G = (V, E)$ consists of a finite vertex set V and an edge set $E \subseteq \binom{V}{2}$.

Before delving deeper, let's establish definitions for various classes of graphs.

1. **Undirected Graph**: An undirected graph is characterized by edges (x, y) being equivalent to (y, x).

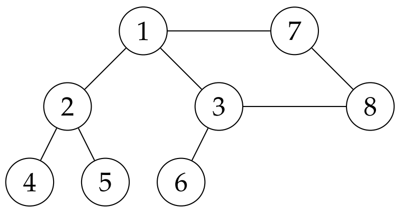

Figure 1.1: Undirected graph

2. **Directed Graph**: A directed graph or digraph $G = (V, E)$ represents edges as ordered pairs of vertices, i.e., $E \subseteq V \times V$.

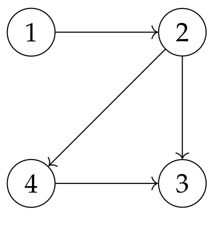

Figure 1.2: Directed graph

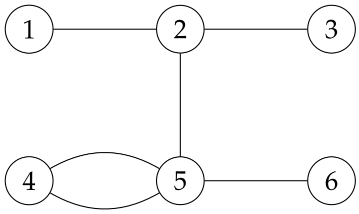

Figure 1.3: Multigraph

3. **Multigraph**: A multigraph $G = (V, E)$ where there can be more than one edge between any given vertices, i.e., $E \subseteq \binom{V}{2}$ is a multiset.

4. **Pseudograph**: A pseudograph $G = (V, E)$ allows loops and multiple edges. Formally, E is a subset of pairs of distinct vertices in V ($\binom{V}{2}$) and pairs with identical elements ($\{(v, v) \mid v \in V\}$), i.e., $E \subseteq \binom{V}{2} \cup \{(v, v) \mid v \in V\}$.

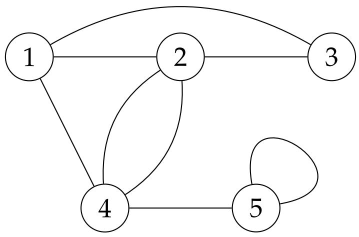

Figure 1.4: Pseudograph

5. **Hypergraph**: A hypergraph $G = (V, E)$ has edges that can be any subset of vertices, expressed as $E \subseteq 2^V$.

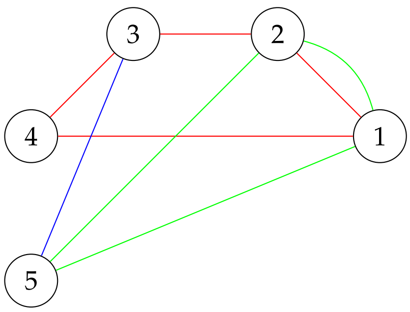

Figure 1.5: Hypergraph $([5], \{\{1, 2, 3, 4\}, \{1, 2, 5\}, \{3, 5\}\})$

6. **Infinite Graph**: A graph where the set V or E is infinite.

1.2 Important Terminologies

Definition 1.2.1 ▶ Order and Size

*The **order** of a graph, denoted by $|V(G)|$, is defined as the cardinality of the vertex set $V(G)$. Similarly, the **size** of the graph, denoted by $|E(G)|$, is defined as the cardinality of the edge set $E(G)$.*

Definition 1.2.2 ▶ Neighbourhood and Degree

*For a given graph G and vertex v, the **neighbourhood** of vertex v, denoted by $N(v)$, is defined as the set of vertices adjacent to v: $N(v) = \{x \in V(G) \mid \{v, x\} \in E(G)\}$. The **degree** of vertex v, denoted as $\deg(v)$, is defined as the cardinality of its neighbourhood: $\deg(v) = |N(v)|$.*

We introduce two more notations:

- The maximum degree of a graph G is denoted by $\Delta(G)$: $\Delta(G) = \max\{\deg(v) \mid v \in V(G)\}$.
- The minimum degree of a graph G is denoted by $\delta(G)$: $\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$.

Now, let's state one of the famous and basic theorems.

Theorem 1.2.1

In a graph G, the sum of the degrees of vertices is equal to twice the number of edges:

$$2|E(G)| = \sum_{v \in V(G)} \deg(v)$$

Proof. Easy exercise. Try to double count the set $\{(v, e) \in V \times E \mid v \in e\}$. $\square$

We will now introduce two very useful concepts: *vertex deletion* and *edge deletion*.
The notation $G - v$ denotes the graph obtained by excluding vertex v and all its incident edges from G, expressing the concept of vertex deletion. This can be formally described as

$$G - v = (V(G) \setminus \{v\}, E(G) \setminus \{e : v \in e\})$$

Similarly, $G - e$ signifies the graph resulting from the removal of a specific edge e in G, while keeping its original end vertices intact. In other words,

$$G - e = (V(G), E(G) \setminus \{e\})$$

Additionally, the notation G/e denotes the graph obtained by merging the end vertices v_1 and v_2 of edge $e = \{v_1, v_2\}$ into a single vertex v. In the graph G, every edge incident on either v_1 or v_2 is now incident on v in the updated notation G/e, which can be formally expressed as

$$G/e = (V(G), E(G))/\sim_e$$

where $\sim_e$ denotes the equivalence relation $v_1 \sim_e v_2$.

We now define the concepts of connectedness and cut sets.

Definition 1.2.3 ▶ Connected Graph

A graph G is connected if, for every pair of vertices u, v in G, there exists a sequence of edges $e_1, \ldots, e_k$ such that $u \in e_1$, $v \in e_k$, and the intersection size $|e_i \cap e_{i+1}| = 1$ for all i.

Definition 1.2.4 ▶ Connected Component

Let $G = (V, E)$ be a graph. A connected component of G is a maximal subgraph $G' = (V', E')$ of G, such that:

- *Every vertex in V' is connected to every other vertex in V' by a path in G'.*
- *There is no vertex in $V - V'$ that can be added to V' without violating the previous condition.*

Definition 1.2.5 ▶ Cut Vertex

Let $G = (V, E)$ be a graph. A vertex v in V is a cut vertex if and only if the removal of v from G results in an increase in the number of connected components of G.

Definition 1.2.6 ▶ Cut Edge or Bridge

Let $G = (V, E)$ be a graph. An edge $e \in E$ is a cut edge or bridge if and only if the removal of e from G results in an increase in the number of connected components of G.

Definition 1.2.7 ▶ Vertex Cut Set

Let $G = (V, E)$ be a graph. A proper subset $S \subset V(G)$ is a vertex cut set if and only if the removal of S from G results in a disconnected graph.

Definition 1.2.8 ▶ Connectivity

Let $G = (V, E)$ be a graph. The connectivity of G, denoted by $\kappa(G)$, is defined as the minimum size of a cut set of G.

Next, we go on to define a few more types of graphs.

1. **Complement of a Graph**: The complement of a graph G is the graph on n vertices, where all possible vertices $(x, y) \notin E(G)$.
2. **Line Graph of a Graph**: The line graph $L(G)$, where each edge in G is represented by a vertex in $L(G)$, and an edge exists between two vertices of $L(G)$ if the corresponding edges in G share a vertex.
3. **Regular Graph**: A graph whose every vertex has equal degree.
4. **Bipartite Graph**: A graph whose vertices can be divided into two partite sets X and Y, such that no two vertices in a partite set are adjacent to each other.
5. **Subgraph**: A graph H is called the subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 1.2.9 ▶ Graph Homomorphism and Isomorphism

Suppose G and H are two graphs. A function $\phi : V(G) \to V(H)$ is a graph homomorphism if $\{x, y\} \in E(G)$ implies $\{\phi(x), \phi(y)\} \in E(H)$. ϕ is an isomorphism if ϕ is a bijection, and $\{x, y\} \in E(G)$ if and only if $\{\phi(x), \phi(y)\} \in E(H)$, i.e., ϕ and ϕ^{-1} are graph homomorphisms. G and H are isomorphic ($G \cong H$) if there exists an isomorphism between G and H. An automorphism is an isomorphism $\phi : G \to G$ from a graph G to itself.

That concludes the basic definitions we will be needing. Further on, we will introduce various interesting ideas, and in our Random Graphs WRP, these ideas will be of essence.

Chapter 2

Distance in Graphs

2.1 Basic Ideas

Distance in graphs are a very important concepts, especially, in respect to real-world networks. Distance functions can be defined as metrics.

Definition 2.1.1 ▶ Metric

A metric on a set X is a function $d : X \times X \to \mathbb{R}$, such that, for all $x, y, z \in X$,

1. $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$.
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Before talking more about distances in graphs, we need to know what a path is. Understanding what a path is, makes it easier for us to discuss the distance between any two vertices in a graph.

Definition 2.1.2 ▶ Walk

A walk in a graph is a sequence of vertices $v_1, v_2, \cdots, v_k$, which are not necessarily distinct, such that $\{v_i, v_{i+1}\} \in E(G)$.

Definition 2.1.3 ▶ Path

A path is a walk, where the vertices are distinct.

As we have already defined what walks and paths are, it would not harm us to define trails too.

Definition 2.1.4 ▶ Trail

A trail is a walk, where the edges are distinct.

We state a theorem that gives the relation between a path and a trail.

Theorem 2.1.1

Every path is a trail but not every trail is a path.

Proof. Easy exercise. □

In a connected graph G, the distance from vertex u to vertex v is the length (number of edges) of a shortest uv path in G. We denote this distance by $d(u, v)$, and in situations where clarity of context is important, we may write $d_G(u, v)$. For a given vertex v of a connected graph, the eccentricity of v, denoted $\mathsf{ecc}(v)$, is defined to be the greatest distance from v to any other vertex. That is,

$$\mathsf{ecc}(v) = \max\{d(v, x) \mid x \in V(G)\}$$

We now define some terminologies related to distances in graphs.

1. The total distance of a vertex $v \in V(G)$ is $d(v) = \sum_{u \in V} d(u, v)$.
2. The average distance of a vertex $v \in V(G)$ is $\bar{d}(v) = \frac{d(v)}{|G|-1}$.
3. The proximity of a graph G is $prox(G) = \min\{\bar{d}(v) \mid v \in V(G)\}$
4. The remoteness of a graph G is $rem(G) = \max\{\bar{d}(v) \mid v \in V(G)\}$
5. The median of a graph, $M(G)$, is the set of vertices v, such that, $\bar{d}(v) = prox(G)$
6. The radius of a graph G is $\mathsf{rad}(G) = \min\{\mathsf{ecc}(v) \mid v \in V(G)\}$
7. The diameter of a graph G is $\mathsf{diam}(G) = \max\{\mathsf{ecc}(v) \mid v \in V(G)\}$
8. The center of a graph $C(G)$, is the set of vertices v, such that, $\mathsf{ecc}(v) = \mathsf{rad}(G)$.
9. The periphery of a graph $P(G)$, is the set of vertices v, such that, $\mathsf{ecc}(v) = \mathsf{diam}(G)$.

The following theorem describes the proper relationship between the radii and diameters of graphs.

Theorem 2.1.2

For any connected graph G, $\mathsf{rad}(G) \leq \mathsf{diam}(G) \leq 2\,\mathsf{rad}(G)$.

Proof. It follows from definition that, $\mathsf{rad}(G) \leq \mathsf{diam}(G)$. Let, u and v be two vertices, such that, $d(u, v) = \mathsf{diam}(G)$. Let, c be a vertex in center of G. Then,

$$\begin{aligned} \mathsf{diam}(G) = d(u, v) &\leq d(u, c) + d(c, v) \\ &\leq 2 \cdot \mathsf{ecc}(c) \\ &= 2 \cdot \mathsf{rad}(G) \end{aligned}$$

This completes the proof. □

2.2 Important Tools

In the context of a single connected component of a disconnected graph, these terms have their normal meanings. If two vertices are in different components, however, we say that the distance between them is infinity. We conclude this chapter with two interesting results.

Theorem 2.2.1

Every graph is isomorphic to the center of some other graph.

Proof. Let, G be any graph. If we add four vertices w, x, y, z to G such that the new graph H is obtained where, $V(H) = V(G) \cup \{w, x, y, z\}$, and $E(H) = E(G) \cup \{wx, yz\} \cup \{xa \mid a \in V(G)\} \cup \{yb \mid b \in V(G)\}$.
Now, $\mathsf{ecc}(w) = \mathsf{ecc}(z) = 4$, $\mathsf{ecc}(y) = \mathsf{ecc}(x) = 3$, and for $v \in V(G), \mathsf{ecc}(v) = 2$. Thus, G is the center of H. □

Theorem 2.2.2

A graph G is isomorphic to the periphery of some graph if and only if either every vertex has eccentricity 1, or no vertex has eccentricity 1.

Proof. Suppose that every vertex of G has eccentricity 1. Not only does this mean that G is complete, it also means that every vertex of G is in the periphery, that is, G is the periphery of itself.

$\Longrightarrow$ On the other hand, suppose that no vertex of G has eccentricity 1. This means that for every vertex u of G, there is some vertex v of G such that $uv \in E(G)$. Now, let H be a new graph, constructed by adding a single vertex, w, to G, together with the edges $wx : x \in V(G)$. In the graph H, the eccentricity of w is 1 (w is adjacent to everything). Further, for any vertex $x \in V(G)$, the eccentricity of x in H is 2 (no vertex of G is adjacent to everything else in G, and everything in G is adjacent to w). Thus, the periphery of H is precisely the vertices of G.

$\Longleftarrow$ Let us suppose that G has some vertices of eccentricity 1 and some vertices of eccentricity greater than 1. Suppose also that G forms the periphery of some graph, say H. Since, the eccentricities of the vertices in G are not all the same, it must be that $V(G)$ is a proper subset of $V(H)$. This means that H is not the periphery of itself and that $\mathsf{diam}(H) \geq 2$. Now, let v be a vertex of G whose eccentricity in G is 1 (v is therefore adjacent to all vertices of G). Since $v \in V(G)$ and since G is the periphery of H, there exists a vertex w in H such that $d(v, w) = \mathsf{diam}(H) \geq 2$. The vertex w, then, is also a peripheral vertex and therefore must be in G. This contradicts the fact that v is adjacent to everything in G.

□

Chapter 3

Paths, Trails, Cycles, and Circuits

We start off this chapter by including some definitions. It would help you all to recall the definitions of paths, walks, and trails.

Definition 3.0.1 ▶ Cycle

A path which ends in the starting vertex is called a closed path or cycle.

Definition 3.0.2 ▶ Circuit

A trail which ends in the starting vertex is called a closed trail or circuit.

Definition 3.0.3 ▶ Eulerin trail

A trail in a graph G that includes every edge of G is called an Eulerian trail.

Definition 3.0.4 ▶ Eulerian circuit

A circuit in a graph G that includes every edge of G is called an Eulerian circuit.

Definition 3.0.5 ▶ Eulerian graph

A graph is said to be Eulerian if it has an Eulerian circuit.

Definition 3.0.6 ▶ Hamiltonian path

If a path P spans the vertices of G, then it is called a Hamiltonian path.

Definition 3.0.7 ▶ Hamiltonian cycle

If a cycle C spans the evrtices of G, then it is called a Hamiltonian cycle.

Definition 3.0.8 ▶ Hamiltonian graph

A graph is said to be Hamiltonian if it has a Hamiltonian cycle.

Theorem 3.0.1

For a connected graph G, TFAE (the following are equivalent) :

(i) G is Eulerian.

(ii) Every vertex of G has even degree.

(iii) The edges of G can be partitioned into edge-disjoint cycles.

The proof of this theorem is intuitive and are left as an exercise for the reader. It will be nice if you try to prove part (iii) and you can send your proofs on the group.

Corollary 3.0.1. *The connected graph G contains an Eulerian trail iff there are at most* 2 *vertices of odd degree.*

Now, we state two very important theorems that will be used both in our Random Graphs WRP as well as in many graph theoretic problems that you may encounter in future.

Theorem 3.0.2 (Dirac's theorem)

Let, G be a graph of order $n \geq 3$*. If* $\delta(G) \geq \frac{n}{2}$*, then G is Hamiltonian.*

Proof. Suppose G is a counterexample to the theorem and G be such a graph with maximal number of edges i.e., addition of an edge to G creates a cycle. Let $v \nsim w$ and hence $G \cup (v, w)$ will contain a Hamilton cycle $v = v_1v_2 \ldots v_n = w, v$. Thus $v_1v_2 \ldots v_n$ is a simple path. Define sets $S_v := \{i : v \sim v_{i+1}\}$ and $S_w := \{i : w \sim v_i\}$. Since $\delta(G) \geq n/2$, $|S_v|, |S_w| \geq n/2$ and further $S_v, S_w \subset \{1, \ldots, n-1\}$. Hence $S_v \cap S_w \neq \emptyset$ and assume that $i_0 \in S_v \cap S_w$. Then $v = v_1v_2 \ldots v_{i_0}w = v_nv_{n-1} \ldots v_{i_0+1}v_1 = v$ is a Hamiltonian circuit in G, contradicting our assumption. □

Theorem 3.0.3 (Ore's theorem)

Let, G be a graph of order $n \geq 3$*. If* $deg(x) + deg(y) \geq n$*, then* $\forall$ *pairs of non-adjacent vertices* x, y*, then G is Hamiltonian.*

The proof of this theorem is left as an exercise to the reader. You can make use of Dirac's theorem as well as the approach used in it's proof to prove Ore's theorem. We encourage you to share your proof in the group.
Next, we define some very useful terminologies that will be of essence.

Definition 3.0.9 ▶ Independent sets and covers

An independent set of vertices *is* $S \subset V$ *such that no two vertices in* S *are adjacent. An* independent set of edges *is* $E' \subset E$ *such that no two edges in* E' *share a common end vertex. A subset of vertices* $S \subset V$ *is a* vertex cover *if every edge in* G *is incident to at least one vertex in* S*. An* edge cover *is a set of edges* $E' \subset E$ *such that every vertex is contained in at least one edge in* E'*.*

Definition 3.0.10 ▶ Independence number and cover number

$$\begin{aligned} \alpha(G) &= \max\{|S| : S \textit{ independent vertex set}\}. \\ \alpha'(G) &= \max\{|M| : M \textit{ independent edge set}\}. \\ \beta(G) &= \min\{|S| : S \textit{ vertex cover}\}. \\ \beta'(G) &= \min\{|E'| : E' \textit{ edge cover}\}. \end{aligned}$$

Now we will state a very interesting theorem which relates the Hamiltonicity of a graph with it's connectivity and independence number.

Theorem 3.0.4

Let, G be a connected graph of order $n \geq 3$ with vertex connectivity $\kappa(G)$ and independence number $\alpha(G)$. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.

You can skip the proof of this theorem if you want, as the proof would not be a crucial part of our WRP.

Proof. If G is as described, then $\kappa(G) \geq 2$, as if $\kappa(G) = 1$, then $\alpha(G) = 1$ and thus G is either K_1 or K_2, contradicting the fact that $n \geq 3$.
Let C be a longest cycle in G. Suppose that C is not a Hamiltonian cycle, and let v be a vertex of G that is not on C. Let, H be the connected component of $G - V(C)$ containing vertex v.
$V(C) = \{c_1, c_2, \ldots, c_r\}$
The vertex of H adjacent to c_i is denoted by h_i. The vertex which is the immediate clockwise successor of c_i is denoted by d_i.
Following these, we can make some observations:

(i) It must be that $r \geq \kappa(G)$. If the vertices $V(C)$ were removed from G, then H would be disconnected from the rest of the graph. Since $\kappa(G)$ is the size of the smallest cut set, it follows that $r \geq \kappa(G) \geq 2$.

(ii) No two of the vertices in the set $V(C)$ are consecutive vertices on C. To see this, suppose that there is some i such that c_i and c_{i+1} are consecutive vertices on C. Let, P be a path from h_i to h_{i+1} in H, and consider the cycle formed by replacing the edge c_ic_{i+1} on C with the path $c_i, [h_i, h_{i+1}]_P, c_{i+1}$. This cycle is longer than our maximal cycle C, a contradiction. This observation implies that the sets $c_1, c_2, \ldots, c_r$ and $d_1, d_2, \ldots, d_r$ are disjoint.

(iii) For each $i (1 \leq i \leq r)$, d_i is not adjacent to v. To see this, suppose $d_iv \in E(G)$ for some i, and let Q be a path from h_i to v in H. In this case, the cycle formed by replacing the edge c_id_i on C with the path $c_i, [h_i, v]_Q, d_i$ is longer than C, again a contradiction.

Now, let $S = v, d_1, d_2, ..., d_r$. The first observation above implies that $|S| \geq \kappa(G) + 1 > \alpha(G)$. This means that some pair of vertices in S must be adjacent. Our third observation implies that d_i must be adjacent to d_j for some $i < j$. If R is a path from h_i to h_j in H, then the cycle $c_i, [h_i, h_j]_R, [c_j, d_i]_{C^-}, [dj, ci]_{C^+}$ is a longer cycle than C. Our assumption that C was not a Hamiltonian cycle has led to a contradiction and thus, C is indeed a Hamiltonian cycle. □

Next, we introduce the idea of forbidden subgraphs. The absence of any particular forbidden subgraph H in a graph G, gives G some "nice" properties. This concept of forbidden subgraphs will be very crucial in understanding and checking for planarity of graphs. Here, we will relate forbidden subgraphs to Hamiltonicity. For the next two theorems, you may skip the proofs if you wish.

Theorem 3.0.5

If G is a 2-connected, $\{K_{1,3}, Z_1\}$-free graph, then G is Hamiltonian

Proof. Suppose G is 2-connected and $K_{1,3}, Z_1$-free, and let C be a longest cycle in G. If C is not a Hamiltonian cycle, then there must exist a vertex v, not on C, which is adjacent to a vertex, say w, on C. Let a and b be the immediate predecessor and successor of w on C.
A longer cycle would exist if either a or b were adjacent to v, and so it must be that both a and b are nonadjacent to v. Now, if a is not adjacent to b, then the subgraph induced by the vertices w, v, a, b is $K_{1,3}$, and we know that G is $K_{1,3}$-free. So it must be that $ab \in E(G)$. But if this is the case, then the subgraph induced by w, v, a, b is Z_1, a contradiction. Therefore, it must be that C is a Hamiltonian cycle. □

Theorem 3.0.6

Let, G be a $\{K_{1,3}, N\}$-free graph. Then:

1. *If G is connected, then G is traceable.*
2. *If G is 2-connected, then G is Hamiltonian.*

The proof of the second part follows directly from theorem 3. The proof of the first part is also simple and the only obstacle that may arise is understanding what we mean by traceable. A traceable graph refers to a graph that has a Hamiltonian path.

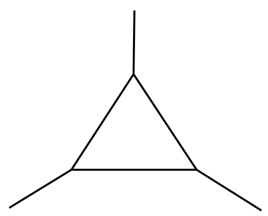

Figure 3.1: This graph is represented by **N**

Theorem 3.0.7 (Mantel's theorem, 1907)

If G is a graph on n vertices with no triangle then $|E| \leq \lfloor n^2/4 \rfloor$. Equivalently, if $|E| > n^2/4$, then $g(G) = 3$.

Proof. Let us divide the problem into 2 cases, $n = 2k$ and $n = 2k + 1$.

Case 1: For n=2k, $\lfloor n^2/4 \rfloor = k^2$.
The theorem holds true for $k = 2$ or $n = 4$. Let, the theorem be true for all $k \leq q$.
Now, we consider $k = q + 1$ or $n = 2q + 2$. Let, the vertex set be V and we consider 2 vertices $x, y \in V$, such that, $x \sim y$. If among the vertices $V \setminus \{x, y\}$, we have more than q^2 edges then a triangle exists and we are done. Let, us assume that there exists no triangle among these vertices $V \setminus \{x, y\}$, then there are atmost q^2 edges among these vertices. Now, we have $(q + 1)^2 + 1 - q^2 - 1 = 2q + 1$ edges more to be put to use. These edges are drawn from either of x or y to the vertices $V \setminus \{x, y\}$. Thus, there exists atleast one vertex $v \in V \setminus \{x, y\}$, such that, $x \sim v$ and $y \sim v$ and thus we get our triangle.

Case 2: For n=2k+1, $\lfloor n^2/4 \rfloor = k^2 + k$.
The theorem holds true for $k = 1$ or $n = 3$. Let, the theorem be true for all $k \leq q$.
Now, we consider $k = q + 1$ or $n = 2q + 3$. Let, the vertex set be V and we consider 2 vertices $x, y \in V$, such that, $x \sim y$. If among the vertices $V \setminus \{x, y\}$, we have more than $q^2 + q$ edges then a triangle exists and we are done. Let, us assume that there exists no triangle among these vertices $V \setminus \{x, y\}$, then there are atmost $q^2 + q$ edges among these vertices. Now, we have at least $(q + 1)^2 + (q + 1) + 1 - q^2 - q - 1 = 2q + 2$ edges more to be put to use. These edges are drawn from either of x or y to the vertices $V \setminus \{x, y\}$. Thus, there exists atleast one vertex $v \in V \setminus \{x, y\}$, such that, $x \sim v$ and $y \sim v$ and thus we get our triangle.

□

Theorem 3.0.8 (Turan, 1941)

If a simple graph on n vertices has no complete subgraph K_p, then $|E| \leq T(n, k) := \frac{(k-2)n^2 - r(k-1-r)}{2(k-1)}$ where $r \equiv n (\mod k - 1)$.

Proof. Let t be such that $n = t(k - 1) + r$. We will prove by induction on t. If $t = 0$, then $n = r, T(n, k) = n(n - 1)/2$ and the theorem trivially holds as $n \leq k - 1$. Now, consider a graph G on n vertices with no K_k subgraph (i.e., a subgraph isomorphic to K_k) and let G have the maximum number of edges subject to these constraints. Hence, G contains a subgraph H isomorphic to K_{k-1}. If not, one can add an edge to G without creating a K_k subgraph and so contradicting its maximality. The vertices $V - H$ are joined to at most $k - 2$ vertices in H. Since $|V - H| = n - k + 1 = (t - 1)(k - 1) + r$ and the induced subgraph $\langle V - H \rangle$ also does not contain a K_k subgraph, by induction hypothesis, $|E(\langle V - H \rangle)| \leq T(n - k + 1, k)$. Thus, we have that

$$|E(G)| \leq T(n - k + 1, k) + (n - k + 1)(k - 2) + \binom{k-1}{2}$$

and one can easily verify that the RHS is equal to $T(n, k)$. □

Chapter 4

Trees

4.1 Basics of Trees and Forests

Definition 4.1.1 ▶ Tree, Forest, Leaf

*A **tree** is a connected graph that contains no cycles. A **forest** is a collection of one or more trees. A vertex of degree 1 in a tree is called a **leaf**.*

As in nature, graph-theoretic trees come in many shapes and sizes. They can be thin (P_{10}) or thick ($K_{1,1000}$), tall (P_{1000}) or short (K_1 and K_2). You can check out how these graphs look by looking them up on google.

There are some interesting names that can be assigned to certain graphs. Such as, K_1 is called a stump and K_2 is called a twig. Similarly, any $K_{1,3}$ graph is called a claw. There are more graphs, which have been assigned such interesting names, and the list is just a web search away from you.

Theorem 4.1.1

A tree T of order n has $(n-1)$ edges.

Proof. We use induction to prove this theorem.

Base Case: For $n = 1$ the only tree is the stump (K_1), and it of course has 0 edges.

Induction Hypothesis: Assume that the result is true for all trees of order less than k.

Inductive Step: Let, T be a tree of order k. Choose some edge of T and call it e. Since T is a tree, it must be that $T - e$ is disconnected with two connected components that are trees themselves. Say that these two components of $T - e$ are T_1 and T_2, with orders k_1 and k_2, respectively. Thus, k_1 and k_2 are less than n and $k_1 + k_2 = k$.

Since $k_1 < k$, the theorem is true for T_1. Thus T_1 has $k_1 - 1$ edges. Similarly, T_2 has $k_2 - 1$ edges. Now, since $E(T)$ is the disjoint union of $E(T_1)$, $E(T_2)$, and $\{e\}$, we have

$$\begin{aligned} |E(T)| &= (k_1 - 1) + (k_2 - 1) + 1 \\ &= k_1 + k_2 - 1 \\ &= k - 1 \end{aligned}$$

Hence, a tree T of order n has $n - 1$ edges. □

The next two theorems can be trivially proved by using the idea of the preceding theorem and so are left as an exercise for the reader. But before stating them we need to define some terminology. We denote the number of connected components in a graph G by $\beta_0(G)$.

Theorem 4.1.2

If F is a forest of order n, then F has $(n - \beta_0(G))$ edges.

Proof. Easy exercise. □

Theorem 4.1.3 (Characterization of tree)

For an n-vertex graph G (with $n > 1$), the following are equivalent (and characterize the trees with n vertices).

- *G is connected and has no cycles.*
- *G is connected and has $n - 1$ edges.*
- *G has $n - 1$ edges and no cycles.*
- *For $u, v \in V(G)$, G has exactly one u, v-path.*

Theorem 4.1.4

If T is a tree of order $n \geq 2$, then T has atleast two leaves.

Proof. Use Theorem 1.2.1 and Theorem 4.1.3 to prove this. □

4.2 Subgraph of Trees or Trees as Subgraphs

Theorem 4.2.1

In any tree, the center is either a single vertex or a pair of adjacent vertices.

Proof. Given a tree T, we form a sequence of trees as follows. Let $T_0 = T$. Let, T_1 be the graph obtained from T_0 by deleting all of its leaves. Note here that T_1 is also a tree. Let, T_2 be the tree obtained from T_1 by deleting all of the leaves of T_1. In general, for as long as it is possible, let T_j be the tree obtained by deleting all of the leaves of T_{j-1}. Since T is finite, there must be an integer r such that T_r is either K_1 or K_2.
Consider now a consecutive pair T_i, T_{i+1} of trees from the sequence $T = T_0, T_1, \ldots, T_r$. Let, v be a non-leaf of T_i. In T_i, the vertices that are at the greatest distance from v are leaves of T_i. This means that the eccentricity of v in T_{i+1} is one less than the eccentricity of v in T_i. Since this is true for all non-leaves of T_i, it must be that the center of T_{i+1} is exactly the same as the center of T_i.
Therefore, the center of T_r is the center of T_{r-1}, which is the center of $T_{r-2}, \ldots$, which is the center of $T_0 = T$. Since, T_r is either K_1 or K_2, the proof is complete. □

We will now add consider trees as subgraphs of graphs, that may or may not be trees.

Theorem 4.2.2

Let T be a tree with k edges. If G is a graph whose minimum degree satisfies $\delta(G) \geq k$, then G contains T as a subgraph. Alternatively, G contains every tree of order at most $\delta(G) + 1$ as a subgraph.

Proof. We induct on k. If $k = 0$, then $T = K_1$, and it is clear that K_1 is a subgraph of any graph. Further, if $k = 1$, then $T = K_2$, and K_2 is a subgraph of any graph whose minimum degree is 1. Assume that the result is true for all trees with $k - 1$ edges ($k \geq 2$), and consider a tree T with exactly k edges. We know that T contains at least two leaves. Let, v be one of them, and let, w be the vertex that is adjacent to v. Consider the graph $T - v$. Since $T - v$ has $k - 1$ edges, the induction hypothesis applies, so $T - v$ is a subgraph of G.
We can think of $T - v$ as actually sitting inside of G. Now, since G contains at least $k + 1$ vertices and $T - v$ contains k vertices, there exist vertices of G that are not a part of the subgraph $T - v$. Further, since the degree in G of w is at least k, there must be a vertex u not in $T - v$ that is adjacent to w. The subgraph $T - v$ together with u forms the tree T as a subgraph of G. □

Definition 4.2.1 ▶ Spanning Tree

Given a graph G, and a subgraph T, we say that T is a spanning tree of G, if T is a tree that contains every vertex of G.

Theorem 4.2.3 (Cayley's Tree formula)

There are n^{n-2} distinct labeled trees of order n. In other words, the number of spanning trees of K_n is n^{n-2}.

Chapter 5

Matchings

5.1 Matchings

Definition 5.1.1 ▶ Matching

A subset M of edges is said to be ***matching*** *if no two edges are incident on any vertex or equivalently, every vertex is contained in at most one edge. A complete matching M on a subset $S \subset V$ is a matching that contains all the vertices in S. A* ***perfect matching*** *is a complete matching on G.*

Alternatively one can consider a matching of a graph M as a subgraph of G such that $d_M(v) = 1$ for all $v \in V(M)$. A matching is perfect if M is spanning. A vertex v is said to be *saturated* if $v \in M$ and else *unsaturated*. For a subset $S \subset V$, $N(S) = \bigcup_{v \in S} N(v)$.

Theorem 5.1.1 (Hall's marriage theorem)

Let G be a bi-partite graph with the two vertex sets being V_1, V_2. Then there exists a complete matching on V_1 iff $|N(S)| \geq |S|$ for all $S \subset V_1$.

Proof. Let $|V_1| = k$ and our proof will be by induction on k. If $k = 1$, the proof is trivial. Let $G = V_1 \cup V_2$ be such that the result holds for any graph with strictly smaller V_1.
Suppose that $|N(S)| \geq |S| + 1$ for all $S \subsetneq V_1$. Then choose $(v, w) \in E \cap V_1 \times V_2$ and consider the induced subgraph $G' := \langle V - \{v, w\} \rangle$. Since we have removed only w from V_2 and that $|N(S)| \geq |S| + 1$ for all $S \subsetneq V_1$, we get that $|N(S')| \geq |S'|$ for all $S' \subset V_1 - \{v\}$. Thus there is a complete matching M on $V_1 - \{v\}$ in G' by induction hypothesis and $M \cup (v, w)$ is a complete matching on V_1 in G as desired.
If the above is not true i.e., there exists $A \subset V_1$ such that $N(A) = B$ and $|A| = |B|$. Then, by induction hypothesis, there is a complete matching M_0 on A in the induced subgraph $\langle A \cup B \rangle$. Trivially, Hall's condition holds i.e., for all $S \subset A$, $|N(S) \cap B| = |N(S)| \geq |S|$. Let $G' := G - \angle A \cup B \rangle$. Let $S \subset V_1 - A$. Suppose if $|N'(S)| < |S|$ where $N'(S) = N(S) \cap (V_2 - B)$. Then, we have that $N(S \cup A) = N'(S) \cup B$ and hence $|N(S \cup A)| \leq |N'(S)| + |B| < |S| + |A|$, a contradiction. Hence, G' also satisfies Hall's condition and again by induction hypothesis G' has a complete matching M' on $V_1 - A$. Thus, we have a complete matching $M := M_0 \cup M'$ on V_1 in G. $\square$

Proposition 5.1.1. *Let $d \geq 1$. Let G be a bipartite graph on $V_1 \sqcup V_2$ such that $|N(S)| \geq |S| - d$ for all $S \subset V_1$. Then G has a matching with at least $|V_1| - d$ independent edges.*

Proof. Set $V_2' := V_2 \cup [d]$. Define G' with vertex set as $V_1 \sqcup V_2'$ and edge set as $E(G) \cup (V_1 \times [d])$. Then, it is easy to see that Hall's condition is true on G' and hence there is a complete matching M of V_1 in G'. Now, if we remove the edes in M incident on $[d]$, we get a matching with at least $|V_1| - d$ edges as required. □

Definition 5.1.2 ▶ Factor of a graph

Given a graph G, a factor of G is a spanning subgraph. Equivalently, a subgraph H is said to be a factor (of G) if $V(H) = V(G)$. An r-factor is a factor that is r-regular.

Thus, 1-factors are nothing but perfect matchings.

Theorem 5.1.2 (Petersen, 1891)

Every regular graph of positive even degree has a 2-factor.

Proof. Easy exercise. □

A matching M is said to be *maximal* if there is no matching M' such that $M \subsetneq M'$. A matching M is said to be *a maximum matching* if $\alpha'(G) = |M|$.
Now recall the definitions of $\alpha(G), \beta(G), \alpha'(G), \beta'(G)$. If M is a maximum matching, then to cover each edge we need distinct vertices and hence the vertex cover should have size at least $|M|$. Furthermore, given a maximum matching M, $V(M)$ gives a vertex cover. For if there is an edge e not covered by $V(M)$ then $M + e$ is a larger matching than M. These observations yield the first inequality below.

$$\alpha'(G) \le \beta(G) \le 2\alpha'(G) \text{ and } \alpha(G) \le \beta'(G)$$

As for the second inequality, observe that to cover vertices of an independent set, we need distinct edges.

Lemma 5.1.2. *Let G be a graph. $S \subset V$ is an independent set iff S^c is a vertex cover. As a corollary, we get $\alpha(G) + \beta(G) = n = |V|$.*

Theorem 5.1.3 (Konig-Egervary theorem)

For a bi-partite graph, $\alpha'(G) = \beta(G)$.

Proof. We will show that for a minimum vertex cover Q, there exists a matching of size at least $|Q|$. Partition Q into $A := Q \cap V_1$ and $B := Q \cap V_2$. Let H and H' be induced subgraphs on $A \sqcup (V_2 - B)$ and $(V_1 - A) \sqcup B$ respectively. If we show that there is a complete matching on A in H and a complete matching on B in H', we have a matching of size at least $|A| + |B|$ $(= |Q|)$ in G. Also, note that it suffices to show that there is a complete matching on A in H because we can reverse the roles of A and B apply the same argument to B as well.
Since $A \cup B$ is a vertex cover, there cannot be an edge between $V_1 - A$ and $V_2 - B$. Suppose for some $S \subset A$, we have that $|N_H(S)| < |S|$. Since $N_H(S)$ covers all edges from S that are not incident on B, $Q' := Q - S + N_H(S)$ is also a vertex cover. By choice of S, Q' is a smaller vertex cover than Q contradicting that Q is minimum. Hence, we have that Hall's condition holds true for A in H. And by the arguments in the previous paragraph, the proof is complete. □

Theorem 5.1.4 (Gallai, 1959)

If G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n = |V|$.

Proof. Suppose M is a maximum matching. Then $S = V - V(M)$ is also an independent set. If there are edges between vertices of S, then such edges can be added to M and one can obtain a larger matching. Hence there are no edges between vertices of S and hence it is a independent set. Construct a edge cover as follows : Add all edges in M to Q and for each $v \in S$, add one of its adjacent edges to Q. Since there are no isolated vertices, v has atleast one adjacent edge. Thus $|Q| = |M| + |S|$ and since $V(M) \sqcup S = V$, we can derive that

$$\alpha'(G) + \beta'(G) \leq |M| + |Q| = 2|M| + |S| = n$$

Let Q be a minimum edge cover. Then Q cannot contain a path of length more than 2. Else, by removing the middle edge in a path of length at least 3, we can obtain a smaller edge cover. By the previous exercise, Q is a graph consisting of star components. If $C_1, \ldots, C_k$ are the components of Q, then $V(C_1) \cup \ldots \cup V(C_k) = V$ and $E(C_1) \cup \ldots \cup E(C_k) = Q$. Now choose a matching $M = \{e_1, \ldots, e_k\}$ by selecting one edge from every component $C_1, \ldots, C_k$. Since C_i's are disjoint, M is a matching. Thus, using the fact that each Q is a forest with k components, we can derive that

$$\alpha'(G) + \beta'(G) \geq |M| + |Q| = k + |E(Q)| = n$$

□

As a corollary, we get König's result : if G is bi-partite graph without isolated vertices, $\alpha(G) = \beta'(G)$.

5.2 Augmenting path

Definition 5.2.1 ▶ Augmenting path

Given a matching M, a M-alternating path P is a path such that its edges alternate between M and M^c. A M-augmenting path is a M-alternating path whose end-vertices do not belong to M.

Theorem 5.2.1 (Berge, 1957)

A matching M in a graph is a maximum matching in G iff G has no M-augmenting path.

Proof. Suppose there is an M-augmenting path P. Let $P = v_0 v_1 \ldots v_k$. Since P is M-augmenting, $(v_0, v_1), (v_2, v_3), \ldots, (v_{k-1}, v_k) \notin M$ and $(v_1, v_2), (v_3, v_4), \ldots, (v_{k-2}, v_{k-1}) \in M$. Now, observe that $M' = M - P \cup \{(v_0, v_1), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$ is a larger matching than M. Hence if M is a maximum matching, there is no M-augmenting path. Suppose M' is a larger matching than M. We shall construct an M-augmenting path and prove the theorem by contraposition. Let $F = M \triangle M'$. We know by the above exercise that the components of F are paths or even cycles. Since $|M'| > |M|$, there must be a component of F such that M' has more edges in that component than M'. If a component in F is an even cycle, it consists of same number of edges from M and M'. Thus, the component for which M' has more edges must be a path, say $P = v_0 \ldots, v_k$. Since $P \subset F$, we have

Figure 5.1: Red is a maximal matching and Blue is a maximum matching

that P has to be an M-alternating path i.e., $(v_0, v_1) \in M', (v_1, v_2) \in M, \dots$ or $(v_0, v_1) \in M, (v_1, v_2) \in M', \dots$. Since $m' := |M' \cap P| > |M \cap P| = m$ and that P is an M-alternating path, we derive that $m' - m = 1$ and $k = 2m + 1$. Further, this implies that $(v_0, v_1), (v_2, v_3), \dots, (v_{k-1}, v_k) \in M'$ and $(v_1, v_2), (v_3, v_4), \dots, (v_{k-2}, v_{k-1}) \in M$ i.e., P is an M-alternating path. If $v_0 \in V(M)$ then there exists $(w, v_0) \in M$ for some $w \neq v_1$. Also $(w, v_0) \in M \setminus M' \subset F$ contradicting the assumption that P is not a component. So $v_0 \notin M$ and similarly $v_k \notin M$. Thus, we have that P is an M-augmenting path as needed. $\square$

Recall definition of graph factors. For a graph G, let $o(G)$ denote the number of odd components of G. The next theorem that we will be stating has a long proof and you may skip it if you want.

Theorem 5.2.2 (Tutte's 1-factor theorem)

A graph has a 1-factor iff $o(G - S) \leq |S|$ for all $S \subset V$.

Proof. ***To Be Added...*** $\square$

We end this chapter by stating Menger's theorem. However, before diving into the theorem we need to look at some definitions.

Let G be a connected graph, and let u and v be vertices of G. If S is a subset of vertices that does not include u or v, and if the graph $G - S$ has u and v in different connected components, then we say that S is a u, v-separating set.

Theorem 5.2.3

Let G be a graph and let u and v be vertices of G. The maximum number of internally disjoint paths from u to v equals the minimum number of vertices in a u, v-separating set.

You can skip the proof of this theorem if you wish.

Proof. ***To Be Added...*** $\square$

Chapter 6

Planarity

A graph G is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if at all. If G has no such representation, G is called nonplanar. A drawing of a planar graph G in the plane in which edges intersect only at vertices is called a planar representation (or a planar embedding) of G. More precisely,

Definition 6.0.1 ▶ Planar Graph

A graph $G = (V, E)$ is called a planar graph if there exists an injection $\phi : V \to \mathbb{R}^2$, and corresponding to each edge $\{i, j\} \in E$ there exists a continuous curve $\gamma_{ij} : [0, 1] \to \mathbb{R}^2$ such that

$$\phi(i) = \gamma(0) \text{ and } \phi(j) = \gamma(1)$$

and for any two edges $e_1, e_2 \in E$

$$\gamma_{e_1}([0, 1]) \cap \gamma_{e_2}([0, 1]) = \phi(e_1 \cap e_2)$$

having cardinality atmost 2.

If three or more edges bound a portion of a graph then we call it a region. In figure 6.1, $R_1, R_2, \ldots, R_7$ are regions in the graph. An edge e bounds a region R, if it comes in contact with R. We denote the bound degree of R by $b(R)$ and define it as the number of edges that bound region R.

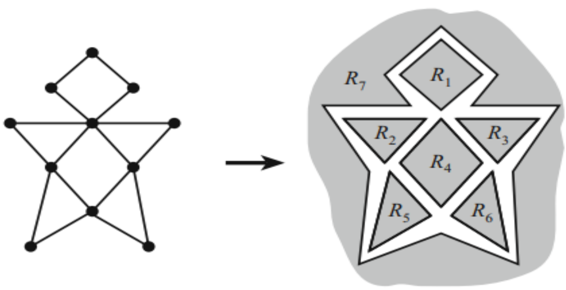

Figure 6.1: Representation of regions in a graph

Now, we are ready to state the famous Euler's formula.

Theorem 6.0.1 (Euler's formula)

For a connected planar graph G with n vertices, q edges, and r regions, then $n - q + r = 2$.

Proof. We induct on q, the number of edges. If $q = 0$, then G must be K_1, a graph with 1 vertex and 1 region. The result holds in this case. Assume that the result is true for all connected planar graphs with fewer than q edges, and assume that G has q edges.
Case 1. Suppose G is a tree. We know from our work with trees that $q = n - 1$, and $r = 1$, since a planar representation of a tree has only one region. Thus, $n - q + r = n - (n - 1) + 1 = 2$, and the result holds.
Case 2. Suppose G is not a tree. Let C be a cycle in G, let e be an edge of C, and consider the graph $G - e$. Compared to G, this graph has the same number of vertices, one edge fewer, and one region fewer, since removing e coalesces two regions in G into one in $G - e$. Thus the induction hypothesis applies, and in $G - e$, $n - (q - 1) + (r - 1) = 2$, implying that $n - q + r = 2$.
The result holds in both cases, and the induction is complete. □

Euler's formula helps us a lot in identifying non-planar graphs. We urge you to check using Euler's formula that $K_{3,3}$ and K_5 are non-planar. A more general general version of the Euler's formula has been stated below.

Theorem 6.0.2

For a planar graph G with n vertices, q edges, and r regions, then $n - q + r = 1 + \beta_0(G)$.

Theorem 6.0.3

If G is a planar graph with $n \geq 3$ vertices and q edges, then $q \leq 3n - 6$. Furthermore, if equality holds, then every region is bounded by three edges.

Proof. Let, us consider $C = \sum_R b(R)$.
Since every edge of G is shared by at most 2 regions so, $C \leq 2q$. Further as each region is bounded by atleast 3 edges, so $C \geq 3r$. Thus,

$$\begin{aligned}
&3r \leq 2q \\
\Longrightarrow &3(2 + q - n) \leq 2q \\
\Longrightarrow &6 + 3q - 3n \leq 2q \\
\Longrightarrow &q \leq 3n - 6
\end{aligned}$$

If equality holds, then $3r = 2q$, and it must be that every region is bounded by three edges. □

Theorem 6.0.4

If G is a planar graph, then $\delta(G) \leq 5$.

Proof. Suppose G has n vertices and q edges. If $n \leq 6$, then the result is immediate, so we will suppose that $n > 6$. If we let D be the sum of the degrees of the vertices of G, then we have:

$$D = 2q \leq 2(3n - 6) = 6n - 12$$

If each vertex had degree 6 or more, then we would have $D \geq 6n$, which is impossible. Thus there must be some vertex with degree less than or equal to 5. $\square$

Now, we define what we mean by a subdivision as the concluding portion of this chapter will cover two theorems that are very important and will go a long way in helping us outright identify many graphs as non-planar.

Definition 6.0.2

A subdivision of an edge G is a substitution of a path for e.

Definition 6.0.3

We say that, H is a subdivision of G if H can be obtained from G by a finite sequence of subdivisions.

Now that the idea of subdivisions has been introduced we can finally move on to the last two theorems for this chapter.

Theorem 6.0.5

A graph G is planar iff every subdivision of G is planar.

The proof for this is very intuitive as so is left as an exercise for the reader.
Lastly, we end this chapter by stating and proving Kuratowski's theorem.

Theorem 6.0.6

A graph G is planar iff it contains no subdivision of $K_{3,3}$ or K_5.

We state this theorem without proof as the proof is not very easy. However, you acn check out the proof of Kuratowski's theorem online.

Chapter 7

Colorings

Given a graph G and a positive integer k, a k-coloring is a function $K : V(G) \to \{1, \ldots, k\}$ from the vertex set into the set of positive integers less than or equal to k. If we think of the latter set as a set of k "colors," then K is an assignment of one color to each vertex. We say that K is a proper k-coloring of G if for every pair u, v of adjacent vertices, $K(u) \neq K(v)$. If such a coloring exists for a graph G, we say that G is k-colorable. Given a graph G, the chromatic number of G, denoted by $\chi(G)$, is the smallest integer k, such that, G is k-colorable.
We first look at the chromatic numbers of some basic graphs.

- $\chi(C_n) = 2$ if n is even and $\chi(C_n) = 3$ if n is odd
- $\chi(K_n) = n$ and $\chi(K_{m,n}) = 2$
- $\chi(P_n) = 2$ if $n \geq 2$ and $\chi(P_n) = 1$ if $n = 1$

Theorem 7.0.1

For any graph G of order n, $\chi(G) \leq n$.

The proof of the theorem is intuitive and is left as an exercise for the reader.
Before moving forward, we want to point out what is famously known as the greedy algorithm. It can be used to prove several theorems related to coloring.

Greedy algorithm

(i) Mark all the vertices as $V = \{v_1, v_2, \ldots, v_n\}$.

(ii) Next, order the available colors in some way. We will denote them by the positive integers $1, 2, \ldots, n$. Then start coloring by assigning color 1 to vertex v_1. Next, if v_1 and v_2 are adjacent, assign color 2 to vertex v_2; otherwise, use color 1 again. In general, to color vertex v_i, use the first available color that has not been used for any of v_i's previously colored neighbors.

Theorem 7.0.2

For any graph G, $\chi(G) \leq \Delta(G) + 1$.

Proof. Greedy algorithm uses at most $\Delta(G) + 1$ colors as every vertex in the graph is adjacent to atmost $\Delta(G)$ other vertices, and hence the largest color label used is at most $\Delta(G) + 1$. The equality in the theorem holds only for complete graph and odd cycles. □

Next, we go on to state Brooks's theorem. The proof of the theorem is significantly long and not important for the WRP. So, you are free to skip the proof.

Theorem 7.0.3 (Brooks's theorem)

If G is a connected graph that is neither an odd cycle nor a complete graph, then $\chi(G) \leq \Delta(G)$.

Proof. ***To Be Added...*** □

The next bound involves a new concept.

Definition 7.0.1

The clique number of a graph, denoted by $\omega(G)$, *is defined as the order of the largest complete graph that is a subgraph of* G.

We will be stating the following two bounds on chromatic numbers without proof. The reader is urged to try to prove the theorems by themselves.

Theorem 7.0.4

For any graph G, $\chi(G) \geq \omega(G)$.

Theorem 7.0.5

For any graph G *on* n *vertices,* $\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$.

Next, we will be stating the five-color theorem.

Theorem 7.0.6 (Five Color theorem)

Every planar graph is 5*-colorable.*

Proof. We induct on the order of G. Let G be a planar graph of order n. If $n \leq 5$, then the result is clear. So suppose that $n \geq 6$ and that the result is true for all planar graphs of order $n - 1$. We know that G contains a vertex, say v, having $deg(v) \leq 5$ (as every planar graph has $\delta(G) \leq 5$). Consider the graph G' obtained by removing from G the vertex v and all edges incident with v. Since the order of G' is $n - 1$ (and since G' is of course planar), we can apply the induction hypothesis and conclude that G' is 5-colorable. Now, we can assume that G' has been colored using the five colors, named $1, 2, 3, 4$, and 5. Consider now the neighbors of v in G. As noted earlier, v has at most five neighbors in G, and all of these neighbors are vertices in (the already colored) G'. If in G' fewer than five colors were used to color these neighbors, then we can properly color G by using the coloring for G' on all vertices other than v, and by coloring v with one of the colors that is not used on the neighbors of v. In doing this, we have produced a 5-coloring for G.

So, assume that in G exactly five of the colors were used to color the neighbors of v. This implies that there are exactly five neighbors, call them w_1, w_2, w_3, w_4, w_5, and assume without loss of generality that each wi is colored with color i. We wish to rearrange the colors of G so that we make a color available for v. Consider all of the vertices of G that have been colored with color 1 or with color 3. *Case 1. Suppose that in G there does not exist a path from w_1 to w_3 where all of the colors on the path are* 1 *or* 3*. Define a subgraph H of G to be the union of all paths that start at w_1 and that are colored with either* 1 *or* 3*. Note that w_3 is not a vertex of H and that none of the neighbors of w_3 are in H. Now, interchange the colors in H. That is, change all of the* 1*'s into* 3*'s and all of the* 3*'s into* 1*'s. The resulting coloring of the vertices of G is a proper coloring, because no problems could have possibly arisen in this interchange. We now see that w_1 is colored* 3*, and thus color* 1 *is available to use for v. Thus, G is* 5*-colorable.* Case 2. Suppose that in G there does exist a path from w_1 to w_3 where all of the colors on the path are 1 or 3. Call this path P. Note now that P along with v forms a cycle that encloses either w_2 or w_4. So there does not exist a path from w_2 to w_4 where all of the colors on the path are 2 or 4. Thus, the reasoning in *Case 1 applies. We conclude that G is* 5*-colorable.* □

Following this, we now state the four-color theorem. We won't be including the proof for very obvious reasons. But rest assured it is no more a conjecture and has been proven.

Theorem 7.0.7 (Four Color theorem)

Every planar graph is 4*-colorable.*

Chapter 8

Spectral Graph Theory

This chapter employs linear algebra tools to systematically investigate graph properties, thereby facilitating a more lucid comprehension and expeditious validation of proofs for both antecedent and contemporary results.

We start by clarifying our convention. For a graph $G = (V, E)$ on n vertices and m edges, we denote the vertex set $V(G)$ as $[n] = \{1, \ldots, n\}$ and the edge set $E(G)$ as $E = \{e_1, \ldots, e_m\}$. We will also assume that the vertices of G are ordered in some way.

8.1 Basic facts from Linear Algebra

To be added...

8.2 Incidence Matrix

Definition 8.2.1

Let G be a graph with n vertices and m edges, we first assign an orientation to the edges and consider them as $E(G) = \left\{e_j = \left(e_j^+, e_j^-\right) \mid j = 1, \ldots, m\right\}$. Here, e_j^+ denotes the vertex where the edge e_j is outgoing and e_j^- denotes the vertex where the same is incoming. Then the ***incidence matrix*** *of G is the $n \times m$ matrix $Q = \left((q_{ij})\right)_{m \times n}$ defined as follows:*

$$q_{ij} = \begin{cases} 1 & \text{if } v_i = e_j^+ \\ -1 & \text{if } v_i = e_j^- \\ 0 & \text{otherwise} \end{cases}$$

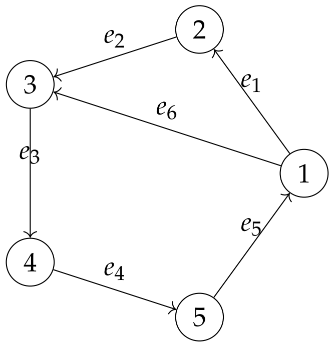

Figure 8.1: Graph $G_1 = ([5], \{e_1, \ldots, e_6\})$

Example 8.2.1

Consider the graph G_1 having the incidence matrix $Q(G_1)$ as follows:

$$Q(G_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

Now, we list some basic properties of the incidence matrix.

Theorem 8.2.1

Consider a graph G with n vertices and m edges. For two incidence matrices Q_1 and Q_2 of G, there exists a diagonal matrix D of order m with diagonal entries ± 1 such that $Q_1 = Q_2 D$.

Proof. Considering the digraphs corresponding to Q_1 and Q_2, and noting their shared underlying graph with identical vertices and edges, we relabel the edges of Q_2 to align with Q_1. Define the diagonal matrix D such that $D_{ii} = 1$ if the i^{th} edge has the same orientation in both Q_1 and Q_2, and $D_{ii} = -1$ if their orientations are opposite. This establishes $Q_1 = Q_2 D$, which completes the proof. □

So, it doesn't matter which orientation we choose for the edges of a graph, because every incidence matrix is the same up to the right multiplication of some ± 1 diagonal matrix.

Next we investigate the rank of the incidence matrix. For a graph G, note that the column sums of $Q(G)$ are zero due to each edge being incident to exactly two vertices, and each vertex being incident to exactly two edges.

Theorem 8.2.2 (Rank)

If G is a connected graph on n vertices, then $\operatorname{rank} Q(G) = n - 1$. More generally, if G has k components, then $\operatorname{rank} Q(G) = n - k$.

Proof. For a connected graph G, let x be in the left null space of $Q := Q(G)$, i.e., $x^T Q = 0$. Since G is connected, all components of x are equal. Thus, the left null space of Q is at most one-dimensional, making the rank of Q at least $n-1$. Also, as the rows of Q are linearly dependent, $\operatorname{rank} Q \le n-1$, implying $\operatorname{rank} Q = n-1$.
If G has k connected components, after relabeling the vertices (if necessary), we can express $Q(G)$ as a block diagonal matrix,

$$Q(G) = \begin{pmatrix} Q(G_1) & & \\ & \ddots & \\ & & Q(G_k) \end{pmatrix}$$

Since each G_i is connected, $\operatorname{rank} Q(G_i) = n_i - 1$, where n_i is the number of vertices in G_i. Thus, $\operatorname{rank} Q = \operatorname{rank} Q_1 + \cdots + \operatorname{rank} Q_k = n - k$. □

Theorem 8.2.3

Let G be a graph on n vertices. Columns $j_1, \ldots, j_k$ of $Q(G)$ are linearly independent if and only if the corresponding edges of G induce an acyclic subgraph.

Proof. Consider edges $j_1, \ldots, j_k$ and suppose there is a cycle in the induced subgraph. Without loss of generality, suppose the columns $j_1, \ldots, j_p$ form a cycle. After relabeling vertices if needed, the submatrix of $Q(G)$ formed by $j_1, \ldots, j_p$ is $\begin{bmatrix} B \\ 0 \end{bmatrix}$, where B is the $p \times p$ incidence matrix of the cycle. Since B is singular (having column sums zero), $j_1, \ldots, j_p$ are dependent, which proves the "only if" part.
Conversely, if $j_1, \ldots, j_k$ induce an acyclic graph (a forest), and the forest has q components, then $k = n - q$, which is the rank of the submatrix formed by $j_1, \ldots, j_k$ (by Theorem 8.2.2). Therefore, the columns $j_1, \ldots, j_k$ are independent. □

Now we look at the square submatrices of the incidence matrix.

Definition 8.2.2 ▶ Totally unimodular matrix

*A matrix is called **totally unimodular** if every square submatrix has determinant $0, 1$, or -1.*

It can be easily proved by induction on the order of the submatrix that $Q(G)$ is totally unimodular which is our next result.

Theorem 8.2.4

Let G be a graph with incidence matrix $Q(G)$. Then $Q(G)$ is totally unimodular.

Proof. We prove the statement that any $k \times k$ submatrix of $Q(G)$ has determinant 0 or ± 1 by induction on k. For $k = 1$, the statement is evident since each entry of $Q(G)$ is either 0 or ± 1. Assuming the statement holds for $k-1$, consider a $k \times k$ submatrix B of $Q(G)$.
If each column of B has a 1 and $k-1$ zeros, or if B has a zero column, then $\det B = 0$. If B has a column with only one nonzero entry, which must be ± 1, expanding the determinant along that column and using the induction assumption implies that $\det B$ must be 0 or ± 1. □

Theorem 8.2.5

Let G be a tree on n vertices. Then any submatrix of $Q(G)$ of order $n-1$ is nonsingular. Moreover, their determinant have the same absolute value.

Proof. Consider any $n-1$ rows of $Q(G)$, say $1, 2, \dots, n-1$, and let B be the submatrix formed by these rows. Let x be a row vector of $n-1$ components in the row null space of B. As in the proof of Theorem 8.2.2, $x_i = 0$ whenever $i \sim n$, and the connectedness of G implies x is the zero vector. Thus, the rank of B is $n-1$, making B nonsingular.

Consider,

$$\det B = \det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix} = \det \begin{pmatrix} -\sum_{j=2}^{n} v_j \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix} = \det \begin{pmatrix} -v_n \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix} = \det \begin{pmatrix} v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}$$

It's your job to convince yourself, no matter which $n-1$ rows we choose, the determinant of the corresponding submatrix remains the same (upto sign). □

8.3 Adjacency Matrix

Definition 8.3.1

*For a graph G with verices $V(G) = [n]$ and edges $E(G) = \{e_1, \dots, e_m\}$, the **adjacency matrix** of G is the $n \times n$ matrix $A = (a_{ij})$ defined as follows:*

$$a_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

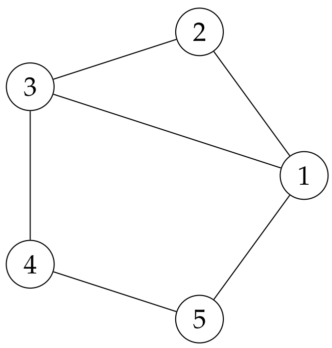

Figure 8.2: Graph G_2

Example 8.3.1

Consider the graph G_2 as undirected version of G_1. It has the adjacency matrix $A(G_2)$ as follows:

$$A(G_2) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Theorem 8.3.1

Let G be a connected graph with vertices $[n]$ and let A be the adjacency matrix of G. The $(i,j)^{th}$ entry a_{ij}^k of A^k counts the number of k-length walks with starting and end vertices i and j respectively.

Proof. By induction on k, the result is evident for $k = 1$. Assuming it holds for $k = m$, consider $A^{m+1} = A^m A$. By induction hypothesis, (i,j)-th entry of A^m counts walks of length m between vertices i and j. Now, the number of walks of length $m+1$ between i and j equals the walks of length m from i to each vertex k adjacent to j. This is expressed as

$$\sum_{k \sim j} a_{ik}^m = \sum_{k=1}^{n} a_{ik}^m a_{kj} = a_{ij}^{m+1}$$

which is precisely the $(i,j)^{\text{th}}$ entry of $A^{m+1} = A^m A$. Hence, the result follows. □

Theorem 8.3.2

Let G be a connected graph with vertices $[n]$ and let A be the adjacency matrix of G. If i, j are vertices of G with $d(i,j) = m$, then the matrices $I, A, \dots, A^m$ are linearly independent.

Proof. We may assume $i \neq j$. There is no (ij)-path of length less than m. Thus, the (i,j)-element of $I, A, \dots, A^{m-1}$ is zero, whereas the (i,j)-element of A^m is nonzero. Hence, the result follows. □

Eigenalues of some graphs

Complete grpah, K_n. $\{n-1, \underbrace{-1, \dots, -1}_{n-1}\}$

Since every vertex of K_n is adjacent to every other vertex, the adjacency matrix consists of all ones except the diagonal entries, which are zero. Thus $A(K_n) = J_n - I_n$, where J_n is the matrix of all ones and I_n is the identity matrix of order n.
Recall that $P_n = \frac{J_n}{n}$ is the projection matrix onto the subspace spanned by the all ones vector. And note that the identity matrix I_n can be decomposed as the direct sum of P_n and $P_n^{\perp}$, which is the projection matrix of it's complement subspace, i.e. $I_n = P_n + P_n^{\perp}$. Thus

$$A(K_n) = J_n - I_n$$

$$= nP_n - (P_n + P_n^{\perp})$$
$$= (n-1)P_n + (-1)P_n^{\perp}$$

which is obviously the spectral decomposition of $A(K_n)$. So the eigenvalues of $A(K_n)$ are $n-1$ with multiplicity $\operatorname{rank} P_n = 1$ and -1 with multiplicity $n - \operatorname{rank} P_n = n-1$.

Complete Bipartite grpah, $K_{p,q}$. $\{\sqrt{pq}, -\sqrt{pq}, \underbrace{0, \ldots, 0}_{p+q-2}\}$.

Note that,

$$A(K_{p,q}) = \begin{pmatrix} 0 & J_{p,q} \\ J_{q,p} & 0 \end{pmatrix}$$

where $J_{p,q}$ is the $p \times q$ matrix of all ones. Since both $J_{p,q}$ and $J_{q,p}$ have rank 1, $\operatorname{rank} A(K_{p,q}) = 2$, yielding 0 as an eigenvalue with multiplicity $p+q-2$. Let λ be a non-zero eigenvalue of A with an eigenvector $v = \begin{pmatrix} x \\ y \end{pmatrix}$. Now, $A(K_{p,q})v = \lambda v$ implies $J_{p,q}y = \lambda x$ and $J_{q,p}x = \lambda y$. So $qJ_p x = \lambda^2 x$. Since J_p has the eigenvalue p with multiplicity 1 and 0 with multiplicity $p-1$, and $\lambda \neq 0$, we get $\lambda^2 = pq$, which gives $\lambda = \pm\sqrt{pq}$. So the eigenvalues of $A(K_{p,q})$ are 0 with multiplicity $p+q-2$ and $\pm\sqrt{pq}$ each with multiplicity 1.

Cycle graph, C_n. $\left\{2\cos\frac{2\pi k}{n} \mid k = 1, \ldots, n\right\}$.

Note that,

$$A(C_n) = U_n + U_n^T$$

where U_n is the upshift matrix of order n, and $U_n U_n^T = I_n = U_n^T U_n$. Utilizing the eigenvalues of U_n, given by $\left\{\omega^k \mid k = 1, \ldots, n\right\}$ where ω is the n^{th} root of unity, we obtain that the eigenvalues of $A(C_n)$ are $\left\{\omega^k + \omega^{n-k} \mid k = 1, \ldots, n\right\}$.

Definition 8.3.2 ▶ Elementary Subgraph

Let G be a graph with $V(G) = [n]$. A subgraph H of G is called an spanning elementary subgraph if each connected component of H is either an edge or a cycle.

We will denote $c(H)$ and $c_1(H)$ as the number of components of H which are edges and cycles respectively.

Theorem 8.3.3 (Determinant)

Let G be a graph with vertices $[n]$ and adjacency matrix A. Then

$$\det A = \sum_{\substack{H \subseteq G \\ \text{spanning} \\ \text{elementary}}} (-1)^{n-c_1(H)-c(H)} 2^{c(H)}$$

8.4 Laplacian

For a graph G with n vertices and m edges, the **Laplacian matrix** of G is the $n \times m$ matrix L defined as $L = QQ^T$

To be added...

Chapter 9

Some Basic Exercises

Chapter 10

****Challenge Problems****

Index