Chapter 1

Basic Definitions

1.1 Notation and Preliminaries

Definition 1.1.1 ▶ Graph

A (simple) graph $G = (V, E)$ consists of a finite vertex set V and an edge set $E \subseteq \binom{V}{2}$.

Before delving deeper, let's establish definitions for various classes of graphs.

1. **Undirected Graph**: An undirected graph is characterized by edges (x, y) being equivalent to (y, x).

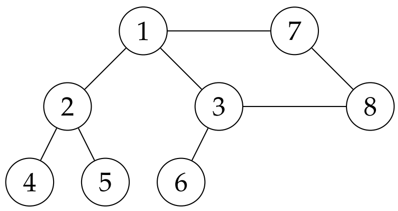

Figure 1.1: Undirected graph

2. **Directed Graph**: A directed graph or digraph $G = (V, E)$ represents edges as ordered pairs of vertices, i.e., $E \subseteq V \times V$.

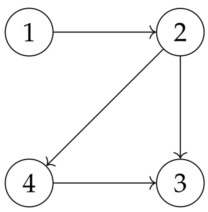

Figure 1.2: Directed graph

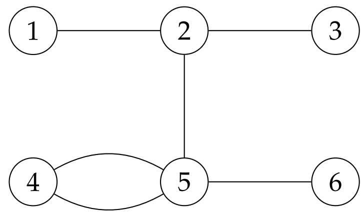

Figure 1.3: Multigraph

3. **Multigraph**: A multigraph $G = (V, E)$ where there can be more than one edge between any given vertices, i.e., $E \subseteq \binom{V}{2}$ is a multiset.

4. **Pseudograph**: A pseudograph $G = (V, E)$ allows loops and multiple edges. Formally, E is a subset of pairs of distinct vertices in V ($\binom{V}{2}$) and pairs with identical elements ($\{(v, v) \mid v \in V\}$), i.e., $E \subseteq \binom{V}{2} \cup \{(v, v) \mid v \in V\}$.

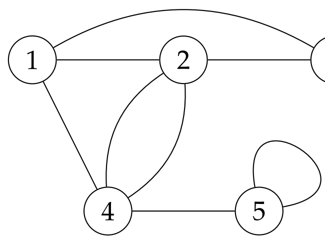

Figure 1.4: Pseudograph

5. **Hypergraph**: A hypergraph $G = (V, E)$ has edges that can be any subset of vertices, expressed as $E \subseteq 2^V$.

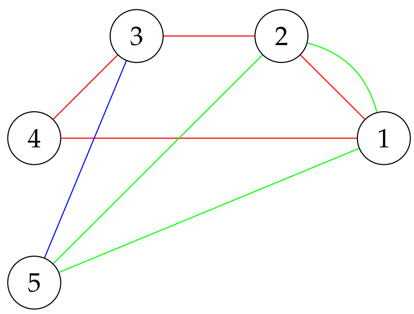

Figure 1.5: Hypergraph $([5], \{\{1, 2, 3, 4\}, \{1, 2, 5\}, \{3, 5\}\})$

6. **Infinite Graph**: A graph where the set V or E is infinite.

1.2 Important Terminologies

Definition 1.2.1 ▶ Order and Size

*The **order** of a graph, denoted by $|V(G)|$, is defined as the cardinality of the vertex set $V(G)$. Similarly, the **size** of the graph, denoted by $|E(G)|$, is defined as the cardinality of the edge set $E(G)$.*

Definition 1.2.2 ▶ Neighbourhood and Degree

*For a given graph G and vertex v, the **neighbourhood** of vertex v, denoted by $N(v)$, is defined as the set of vertices adjacent to v: $N(v) = \{x \in V(G) \mid \{v, x\} \in E(G)\}$. The **degree** of vertex v, denoted as $\deg(v)$, is defined as the cardinality of its neighbourhood: $\deg(v) = |N(v)|$.*

We introduce two more notations:

- The maximum degree of a graph G is denoted by $\Delta(G)$: $\Delta(G) = \max\{\deg(v) \mid v \in V(G)\}$.
- The minimum degree of a graph G is denoted by $\delta(G)$: $\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$.

Now, let's state one of the famous and basic theorems.

Theorem 1.2.1

In a graph G, the sum of the degrees of vertices is equal to twice the number of edges:

$$2|E(G)| = \sum_{v \in V(G)} \deg(v)$$

Proof. Easy exercise. Try to double count the set $\{(v, e) \in V \times E \mid v \in e\}$. □

We will now introduce two very useful concepts: *vertex deletion* and *edge deletion*.
The notation $G - v$ denotes the graph obtained by excluding vertex v and all its incident edges from G, expressing the concept of vertex deletion. This can be formally described as

$$G - v = (V(G) \setminus \{v\}, E(G) \setminus \{e : v \in e\})$$

Similarly, $G - e$ signifies the graph resulting from the removal of a specific edge e in G, while keeping its original end vertices intact. In other words,

$$G - e = (V(G), E(G) \setminus \{e\})$$

Additionally, the notation G/e denotes the graph obtained by merging the end vertices v_1 and v_2 of edge $e = \{v_1, v_2\}$ into a single vertex v. In the graph G, every edge incident on either v_1 or v_2 is now incident on v in the updated notation G/e, which can be formally expressed as

$$G/e = (V(G), E(G))/\sim_e$$

where $\sim_e$ denotes the equivalence relation $v_1 \sim_e v_2$.

We now define the concepts of connectedness and cut sets.

Definition 1.2.3 ▶ Connected Graph

A graph G is connected if, for every pair of vertices u, v in G, there exists a sequence of edges $e_1, \ldots, e_k$ such that $u \in e_1$, $v \in e_k$, and the intersection size $|e_i \cap e_{i+1}| = 1$ for all i.

Definition 1.2.4 ▶ Connected Component

Let $G = (V, E)$ be a graph. A connected component of G is a maximal subgraph $G' = (V', E')$ of G, such that:

- *Every vertex in V' is connected to every other vertex in V' by a path in G'.*
- *There is no vertex in $V - V'$ that can be added to V' without violating the previous condition.*

Definition 1.2.5 ▶ Cut Vertex

Let $G = (V, E)$ be a graph. A vertex v in V is a cut vertex if and only if the removal of v from G results in an increase in the number of connected components of G.

Definition 1.2.6 ▶ Cut Edge or Bridge

Let $G = (V, E)$ be a graph. An edge $e \in E$ is a cut edge or bridge if and only if the removal of e from G results in an increase in the number of connected components of G.

Definition 1.2.7 ▶ Vertex Cut Set

Let $G = (V, E)$ be a graph. A proper subset $S \subset V(G)$ is a vertex cut set if and only if the removal of S from G results in a disconnected graph.

Definition 1.2.8 ▶ Connectivity

Let $G = (V, E)$ be a graph. The connectivity of G, denoted by $\kappa(G)$, is defined as the minimum size of a cut set of G.

Next, we go on to define a few more types of graphs.

1. **Complement of a Graph**: The complement of a graph G is the graph on n vertices, where all possible vertices $(x, y) \notin E(G)$.
2. **Line Graph of a Graph**: The line graph $L(G)$, where each edge in G is represented by a vertex in $L(G)$, and an edge exists between two vertices of $L(G)$ if the corresponding edges in G share a vertex.
3. **Regular Graph**: A graph whose every vertex has equal degree.
4. **Bipartite Graph**: A graph whose vertices can be divided into two partite sets X and Y, such that no two vertices in a partite set are adjacent to each other.
5. **Subgraph**: A graph H is called the subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Definition 1.2.9 ▶ Graph Homomorphism and Isomorphism

Suppose G and H are two graphs. A function $\phi : V(G) \to V(H)$ is a graph homomorphism if $\{x, y\} \in E(G)$ implies $\{\phi(x), \phi(y)\} \in E(H)$. ϕ is an isomorphism if ϕ is a bijection, and $\{x, y\} \in E(G)$ if and only if $\{\phi(x), \phi(y)\} \in E(H)$, i.e., ϕ and ϕ^{-1} are graph homomorphisms. G and H are isomorphic ($G \cong H$) if there exists an isomorphism between G and H. An automorphism is an isomorphism $\phi : G \to G$ from a graph G to itself.

That concludes the basic definitions we will be needing. Further on, we will introduce various interesting ideas, and in our Random Graphs WRP, these ideas will be of essence.