Chapter 1

Matchings

5.1 Matchings

Definition 5.1.1 ▶ Matching

A subset M of edges is said to be ***matching*** *if no two edges are incident on any vertex or equivalently, every vertex is contained in at most one edge. A complete matching M on a subset $S \subset V$ is a matching that contains all the vertices in S. A* ***perfect matching*** *is a complete matching on G.*

Alternatively one can consider a matching of a graph M as a subgraph of G such that $d_M(v) = 1$ for all $v \in V(M)$. A matching is perfect if M is spanning. A vertex v is said to be *saturated* if $v \in M$ and else *unsaturated*. For a subset $S \subset V$, $N(S) = \bigcup_{v \in S} N(v)$.

Theorem 5.1.1 (Hall's marriage theorem)

Let G be a bi-partite graph with the two vertex sets being V_1, V_2. Then there exists a complete matching on V_1 iff $|N(S)| \geq |S|$ for all $S \subset V_1$.

Proof. Let $|V_1| = k$ and our proof will be by induction on k. If $k = 1$, the proof is trivial. Let $G = V_1 \cup V_2$ be such that the result holds for any graph with strictly smaller V_1.
Suppose that $|N(S)| \geq |S| + 1$ for all $S \subsetneq V_1$. Then choose $(v, w) \in E \cap V_1 \times V_2$ and consider the induced subgraph $G' := \langle V - \{v, w\} \rangle$. Since we have removed only w from V_2 and that $|N(S)| \geq |S| + 1$ for all $S \subsetneq V_1$, we get that $|N(S')| \geq |S'|$ for all $S' \subset V_1 - \{v\}$. Thus there is a complete matching M on $V_1 - \{v\}$ in G' by induction hypothesis and $M \cup (v, w)$ is a complete matching on V_1 in G as desired.
If the above is not true i.e., there exists $A \subset V_1$ such that $N(A) = B$ and $|A| = |B|$. Then, by induction hypothesis, there is a complete matching M_0 on A in the induced subgraph $\langle A \cup B \rangle$. Trivially, Hall's condition holds i.e., for all $S \subset A$, $|N(S) \cap B| = |N(S)| \geq |S|$. Let $G' := G - \angle A \cup B \rangle$. Let $S \subset V_1 - A$. Suppose if $|N'(S)| < |S|$ where $N'(S) = N(S) \cap (V_2 - B)$. Then, we have that $N(S \cup A) = N'(S) \cup B$ and hence $|N(S \cup A)| \leq |N'(S)| + |B| < |S| + |A|$, a contradiction. Hence, G' also satisfies Hall's condition and again by induction hypothesis G' has a complete matching M' on $V_1 - A$. Thus, we have a complete matching $M := M_0 \cup M'$ on V_1 in G. □

Proposition 5.1.1. *Let $d \geq 1$. Let G be a bipartite graph on $V_1 \sqcup V_2$ such that $|N(S)| \geq |S| - d$ for all $S \subset V_1$. Then G has a matching with at least $|V_1| - d$ independent edges.*

Proof. Set $V_2' := V_2 \cup [d]$. Define G' with vertex set as $V_1 \sqcup V_2'$ and edge set as $E(G) \cup (V_1 \times [d])$. Then, it is easy to see that Hall's condition is true on G' and hence there is a complete matching M of V_1 in G'. Now, if we remove the edes in M incident on $[d]$, we get a matching with at least $|V_1| - d$ edges as required. □

Definition 5.1.2 ▶ Factor of a graph

Given a graph G, a factor of G is a spanning subgraph. Equivalently, a subgraph H is said to be a factor (of G) if $V(H) = V(G)$. An r-factor is a factor that is r-regular.

Thus, 1-factors are nothing but perfect matchings.

Theorem 5.1.2 (Petersen, 1891)

Every regular graph of positive even degree has a 2-factor.

Proof. Easy exercise. □

A matching M is said to be *maximal* if there is no matching M' such that $M \subsetneq M'$. A matching M is said to be *a maximum matching* if $\alpha'(G) = |M|$.
Now recall the definitions of $\alpha(G), \beta(G), \alpha'(G), \beta'(G)$. If M is a maximum matching, then to cover each edge we need distinct vertices and hence the vertex cover should have size at least $|M|$. Furthermore, given a maximum matching M, $V(M)$ gives a vertex cover. For if there is an edge e not covered by $V(M)$ then $M + e$ is a larger matching than M. These observations yield the first inequality below.

$$\alpha'(G) \leq \beta(G) \leq 2\alpha'(G) \text{ and } \alpha(G) \leq \beta'(G)$$

As for the second inequality, observe that to cover vertices of an independent set, we need distinct edges.

Lemma 5.1.2. *Let G be a graph. $S \subset V$ is an independent set iff S^c is a vertex cover. As a corollary, we get $\alpha(G) + \beta(G) = n = |V|$.*

Theorem 5.1.3 (Konig-Egervary theorem)

For a bi-partite graph, $\alpha'(G) = \beta(G)$.

Proof. We will show that for a minimum vertex cover Q, there exists a matching of size at least $|Q|$. Partition Q into $A := Q \cap V_1$ and $B := Q \cap V_2$. Let H and H' be induced subgraphs on $A \sqcup (V_2 - B)$ and $(V_1 - A) \sqcup B$ respectively. If we show that there is a complete matching on A in H and a complete matching on B in H', we have a matching of size at least $|A| + |B|$ $(= |Q|)$ in G. Also, note that it suffices to show that there is a complete matching on A in H because we can reverse the roles of A and B apply the same argument to B as well.
Since $A \cup B$ is a vertex cover, there cannot be an edge between $V_1 - A$ and $V_2 - B$. Suppose for some $S \subset A$, we have that $|N_H(S)| < |S|$. Since $N_H(S)$ covers all edges from S that are not incident on B, $Q' := Q - S + N_H(S)$ is also a vertex cover. By choice of S, Q' is a smaller vertex cover than Q contradicting that Q is minimum. Hence, we have that Hall's condition holds true for A in H. And by the arguments in the previous paragraph, the proof is complete. □

Theorem 5.1.4 (Gallai, 1959)

If G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n = |V|$.

Proof. Suppose M is a maximum matching. Then $S = V - V(M)$ is also an independent set. If there are edges between vertices of S, then such edges can be added to M and one can obtain a larger matching. Hence there are no edges between vertices of S and hence it is a independent set. Construct a edge cover as follows : Add all edges in M to Q and for each $v \in S$, add one of its adjacent edges to Q. Since there are no isolated vertices, v has atleast one adjacent edge. Thus $|Q| = |M| + |S|$ and since $V(M) \sqcup S = V$, we can derive that

$$\alpha'(G) + \beta'(G) \leq |M| + |Q| = 2|M| + |S| = n$$

Let Q be a minimum edge cover. Then Q cannot contain a path of length more than 2. Else, by removing the middle edge in a path of length at least 3, we can obtain a smaller edge cover. By the previous exercise, Q is a graph consisting of star components. If $C_1, \ldots, C_k$ are the components of Q, then $V(C_1) \cup \ldots \cup V(C_k) = V$ and $E(C_1) \cup \ldots \cup E(C_k) = Q$. Now choose a matching $M = \{e_1, \ldots, e_k\}$ by selecting one edge from every component $C_1, \ldots, C_k$. Since C_i's are disjoint, M is a matching. Thus, using the fact that each Q is a forest with k components, we can derive that

$$\alpha'(G) + \beta'(G) \geq |M| + |Q| = k + |E(Q)| = n$$

$\square$

As a corollary, we get König's result : if G is bi-partite graph without isolated vertices, $\alpha(G) = \beta'(G)$.

5.2 Augmenting path

Definition 5.2.1 ▶ Augmenting path

Given a matching M, a M-alternating path P is a path such that its edges alternate between M and M^c. A M-augmenting path is a M-alternating path whose end-vertices do not belong to M.

Theorem 5.2.1 (Berge, 1957)

A matching M in a graph is a maximum matching in G iff G has no M-augmenting path.

Proof. Suppose there is an M-augmenting path P. Let $P = v_0v_1 \ldots v_k$. Since P is M-augmenting, $(v_0, v_1), (v_2, v_3), \ldots, (v_{k-1}, v_k) \notin M$ and $(v_1, v_2), (v_3, v_4), \ldots, (v_{k-2}, v_{k-1}) \in M$. Now, observe that $M' = M - P \cup \{(v_0, v_1), (v_2, v_3), \ldots, (v_{k-1}, v_k)\}$ is a larger matching than M. Hence if M is a maximum matching, there is no M-augmenting path. Suppose M' is a larger matching than M. We shall construct an M-augmenting path and prove the theorem by contraposition. Let $F = M \triangle M'$. We know by the above exercise that the components of F are paths or even cycles. Since $|M'| > |M|$, there must be a component of F such that M' has more edges in that component than M'. If a component in F is an even cycle, it consists of same number of edges from M and M'. Thus, the component for which M' has more edges must be a path, say $P = v_0 \ldots, v_k$. Since $P \subset F$, we have

Figure 5.1: Red is a maximal matching and Blue is a maximum matching

that P has to be an M-alternating path i.e., $(v_0, v_1) \in M', (v_1, v_2) \in M, \ldots$ or $(v_0, v_1) \in M, (v_1, v_2) \in M', \ldots$. Since $m' := |M' \cap P| > |M \cap P| = m$ and that P is an M-alternating path, we derive that $m' - m = 1$ and $k = 2m + 1$. Further, this implies that $(v_0, v_1), (v_2, v_3), \ldots, (v_{k-1}, v_k) \in M'$ and $(v_1, v_2), (v_3, v_4), \ldots, (v_{k-2}, v_{k-1}) \in M$ i.e., P is an M-alternating path. If $v_0 \in V(M)$ then there exists $(w, v_0) \in M$ for some $w \neq v_1$. Also $(w, v_0) \in M \setminus M' \subset F$ contradicting the assumption that P is not a component. So $v_0 \notin M$ and similarly $v_k \notin M$. Thus, we have that P is an M-augmenting path as needed. □

Recall definition of graph factors. For a graph G, let $o(G)$ denote the number of odd components of G. The next theorem that we will be stating has a long proof and you may skip it if you want.

Theorem 5.2.2 (Tutte's 1-factor theorem)

A graph has a 1-factor iff $o(G - S) \leq |S|$ for all $S \subset V$.

Proof. ***To Be Added...*** □

We end this chapter by stating Menger's theorem. However, before diving into the theorem we need to look at some definitions.

Let G be a connected graph, and let u and v be vertices of G. If S is a subset of vertices that does not include u or v, and if the graph $G - S$ has u and v in different connected components, then we say that S is a u, v-separating set.

Theorem 5.2.3

Let G be a graph and let u and v be vertices of G. The maximum number of internally disjoint paths from u to v equals the minimum number of vertices in a u, v-separating set.

You can skip the proof of this theorem if you wish.

Proof. ***To Be Added...*** □