Chapter 1

Planarity

A graph G is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if at all. If G has no such representation, G is called nonplanar. A drawing of a planar graph G in the plane in which edges intersect only at vertices is called a planar representation (or a planar embedding) of G. More precisely,

Definition 6.0.1 ▶ Planar Graph

A graph $G = (V, E)$ is called a planar graph if there exists an injection $\phi : V \to \mathbb{R}^2$, and corresponding to each edge $\{i, j\} \in E$ there exists a continuous curve $\gamma_{ij} : [0, 1] \to \mathbb{R}^2$ such that

$$\phi(i) = \gamma(0) \text{ and } \phi(j) = \gamma(1)$$

and for any two edges $e_1, e_2 \in E$

$$\gamma_{e_1}([0, 1]) \cap \gamma_{e_2}([0, 1]) = \phi(e_1 \cap e_2)$$

having cardinality atmost 2.

If three or more edges bound a portion of a graph then we call it a region. In figure 6.1, $R_1, R_2, \ldots, R_7$ are regions in the graph. An edge e bounds a region R, if it comes in contact with R. We denote the bound degree of R by $b(R)$ and define it as the number of edges that bound region R.

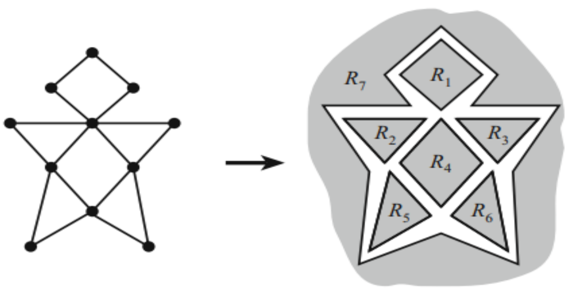

Figure 6.1: Representation of regions in a graph

Now, we are ready to state the famous Euler's formula.

Theorem 6.0.1 (Euler's formula)

For a connected planar graph G with n vertices, q edges, and r regions, then $n - q + r = 2$.

Proof. We induct on q, the number of edges. If $q = 0$, then G must be K_1, a graph with 1 vertex and 1 region. The result holds in this case. Assume that the result is true for all connected planar graphs with fewer than q edges, and assume that G has q edges.
Case 1. Suppose G is a tree. We know from our work with trees that $q = n - 1$, and $r = 1$, since a planar representation of a tree has only one region. Thus, $n - q + r = n - (n - 1) + 1 = 2$, and the result holds.
Case 2. Suppose G is not a tree. Let C be a cycle in G, let e be an edge of C, and consider the graph $G - e$. Compared to G, this graph has the same number of vertices, one edge fewer, and one region fewer, since removing e coalesces two regions in G into one in $G - e$. Thus the induction hypothesis applies, and in $G - e$, $n - (q - 1) + (r - 1) = 2$, implying that $n - q + r = 2$.
The result holds in both cases, and the induction is complete. □

Euler's formula helps us a lot in identifying non-planar graphs. We urge you to check using Euler's formula that $K_{3,3}$ and K_5 are non-planar. A more general general version of the Euler's formula has been stated below.

Theorem 6.0.2

For a planar graph G with n vertices, q edges, and r regions, then $n - q + r = 1 + \beta_0(G)$.

Theorem 6.0.3

If G is a planar graph with $n \geq 3$ vertices and q edges, then $q \leq 3n - 6$. Furthermore, if equality holds, then every region is bounded by three edges.

Proof. Let, us consider $C = \sum_R b(R)$.
Since every edge of G is shared by at most 2 regions so, $C \leq 2q$. Further as each region is bounded by atleast 3 edges, so $C \geq 3r$. Thus,

$$\begin{aligned} & 3r \leq 2q \\ \Longrightarrow & 3(2 + q - n) \leq 2q \\ \Longrightarrow & 6 + 3q - 3n \leq 2q \\ \Longrightarrow & q \leq 3n - 6 \end{aligned}$$

If equality holds, then $3r = 2q$, and it must be that every region is bounded by three edges. □

Theorem 6.0.4

If G is a planar graph, then $\delta(G) \leq 5$.

Proof. Suppose G has n vertices and q edges. If $n \leq 6$, then the result is immediate, so we will suppose that $n > 6$. If we let D be the sum of the degrees of the vertices of G, then we have:

$$D = 2q \leq 2(3n - 6) = 6n - 12$$

If each vertex had degree 6 or more, then we would have $D \geq 6n$, which is impossible. Thus there must be some vertex with degree less than or equal to 5. $\square$

Now, we define what we mean by a subdivision as the concluding portion of this chapter will cover two theorems that are very important and will go a long way in helping us outright identify many graphs as non-planar.

Definition 6.0.2

A subdivision of an edge G is a substitution of a path for e.

Definition 6.0.3

We say that, H is a subdivision of G if H can be obtained from G by a finite sequence of subdivisions.

Now that the idea of subdivisions has been introduced we can finally move on to the last two theorems for this chapter.

Theorem 6.0.5

A graph G is planar iff every subdivision of G is planar.

The proof for this is very intuitive as so is left as an exercise for the reader.
Lastly, we end this chapter by stating and proving Kuratowski's theorem.

Theorem 6.0.6

A graph G is planar iff it contains no subdivision of $K_{3,3}$ *or* K_5.

We state this theorem without proof as the proof is not very easy. However, you acn check out the proof of Kuratowski's theorem online.