Chapter 1

Spectral Graph Theory

This chapter employs linear algebra tools to systematically investigate graph properties, thereby facilitating a more lucid comprehension and expeditious validation of proofs for both antecedent and contemporary results.

We start by clarifying our convention. For a graph $G = (V, E)$ on n vertices and m edges, we denote the vertex set $V(G)$ as $[n] = \{1, \ldots, n\}$ and the edge set $E(G)$ as $E = \{e_1, \ldots, e_m\}$. We will also assume that the vertices of G are ordered in some way.

8.1 Basic facts from Linear Algebra

To be added...

8.2 Incidence Matrix

Definition 8.2.1

*Let G be a graph with n vertices and m edges, we first assign an orientation to the edges and consider them as $E(G) = \left\{e_j = \left(e_j^+, e_j^-\right) \mid j = 1, \ldots, m\right\}$. Here, e_j^+ denotes the vertex where the edge e_j is outgoing and e_j^- denotes the vertex where the same is incoming. Then the **incidence matrix** of G is the $n \times m$ matrix $Q = \left((q_{ij})\right)_{m \times n}$ defined as follows:*

$$q_{ij} = \begin{cases} 1 & \text{if } v_i = e_j^+ \\ -1 & \text{if } v_i = e_j^- \\ 0 & \text{otherwise} \end{cases}$$

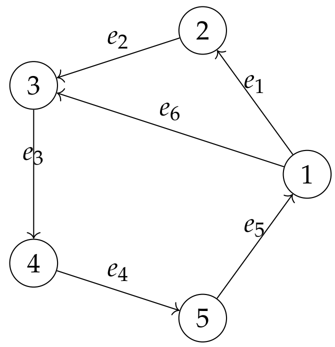

Figure 8.1: Graph $G_1 = ([5], \{e_1, \ldots, e_6\})$

Example 8.2.1

Consider the graph G_1 having the incidence matrix $Q(G_1)$ as follows:

$$Q(G_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix}$$

Now, we list some basic properties of the incidence matrix.

Theorem 8.2.1

Consider a graph G with n vertices and m edges. For two incidence matrices Q_1 and Q_2 of G, there exists a diagonal matrix D of order m with diagonal entries ± 1 such that $Q_1 = Q_2 D$.

Proof. Considering the digraphs corresponding to Q_1 and Q_2, and noting their shared underlying graph with identical vertices and edges, we relabel the edges of Q_2 to align with Q_1. Define the diagonal matrix D such that $D_{ii} = 1$ if the i^{th} edge has the same orientation in both Q_1 and Q_2, and $D_{ii} = -1$ if their orientations are opposite. This establishes $Q_1 = Q_2 D$, which completes the proof. □

So, it doesn't matter which orientation we choose for the edges of a graph, because every incidence matrix is the same up to the right multiplication of some ± 1 diagonal matrix.

Next we investigate the rank of the incidence matrix. For a graph G, note that the column sums of $Q(G)$ are zero due to each edge being incident to exactly two vertices, and each vertex being incident to exactly two edges.

Theorem 8.2.2 (Rank)

If G is a connected graph on n vertices, then $\operatorname{rank} Q(G) = n - 1$. More generally, if G has k components, then $\operatorname{rank} Q(G) = n - k$.

Proof. For a connected graph G, let x be in the left null space of $Q := Q(G)$, i.e., $x^T Q = 0$. Since G is connected, all components of x are equal. Thus, the left null space of Q is at most one-dimensional, making the rank of Q at least $n-1$. Also, as the rows of Q are linearly dependent, $\operatorname{rank} Q \le n-1$, implying $\operatorname{rank} Q = n-1$.
If G has k connected components, after relabeling the vertices (if necessary), we can express $Q(G)$ as a block diagonal matrix,

$$Q(G) = \begin{pmatrix} Q(G_1) & & \\ & \ddots & \\ & & Q(G_k) \end{pmatrix}$$

Since each G_i is connected, $\operatorname{rank} Q(G_i) = n_i - 1$, where n_i is the number of vertices in G_i. Thus, $\operatorname{rank} Q = \operatorname{rank} Q_1 + \cdots + \operatorname{rank} Q_k = n - k$. □

Theorem 8.2.3

Let G be a graph on n vertices. Columns $j_1, \dots, j_k$ of $Q(G)$ are linearly independent if and only if the corresponding edges of G induce an acyclic subgraph.

Proof. Consider edges $j_1, \dots, j_k$ and suppose there is a cycle in the induced subgraph. Without loss of generality, suppose the columns $j_1, \dots, j_p$ form a cycle. After relabeling vertices if needed, the submatrix of $Q(G)$ formed by $j_1, \dots, j_p$ is $\begin{bmatrix} B \\ 0 \end{bmatrix}$, where B is the $p \times p$ incidence matrix of the cycle. Since B is singular (having column sums zero), $j_1, \dots, j_p$ are dependent, which proves the "only if" part.
Conversely, if $j_1, \dots, j_k$ induce an acyclic graph (a forest), and the forest has q components, then $k = n - q$, which is the rank of the submatrix formed by $j_1, \dots, j_k$ (by Theorem 8.2.2). Therefore, the columns $j_1, \dots, j_k$ are independent. □

Now we look at the square submatrices of the incidence matrix.

Definition 8.2.2 ▶ Totally unimodular matrix

A matrix is called ***totally unimodular*** *if every square submatrix has determinant $0, 1$, or -1.*

It can be easily proved by induction on the order of the submatrix that $Q(G)$ is totally unimodular which is our next result.

Theorem 8.2.4

Let G be a graph with incidence matrix $Q(G)$. Then $Q(G)$ is totally unimodular.

Proof. We prove the statement that any $k \times k$ submatrix of $Q(G)$ has determinant 0 or ± 1 by induction on k. For $k = 1$, the statement is evident since each entry of $Q(G)$ is either 0 or ± 1. Assuming the statement holds for $k-1$, consider a $k \times k$ submatrix B of $Q(G)$.
If each column of B has a 1 and $k-1$ zeros, or if B has a zero column, then $\det B = 0$. If B has a column with only one nonzero entry, which must be ± 1, expanding the determinant along that column and using the induction assumption implies that $\det B$ must be 0 or ± 1. □

Theorem 8.2.5

Let G be a tree on n vertices. Then any submatrix of $Q(G)$ of order $n-1$ is nonsingular. Moreover, their determinant have the same absolute value.

Proof. Consider any $n-1$ rows of $Q(G)$, say $1, 2, \ldots, n-1$, and let B be the submatrix formed by these rows. Let x be a row vector of $n-1$ components in the row null space of B. As in the proof of Theorem 8.2.2, $x_i = 0$ whenever $i \sim n$, and the connectedness of G implies x is the zero vector. Thus, the rank of B is $n-1$, making B nonsingular.

Consider,

$$\det B = \det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix} = \det \begin{pmatrix} -\sum_{j=2}^{n} v_j \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix} = \det \begin{pmatrix} -v_n \\ v_2 \\ \vdots \\ v_{n-1} \end{pmatrix} = \det \begin{pmatrix} v_2 \\ \vdots \\ v_{n-1} \\ v_n \end{pmatrix}$$

It's your job to convince yourself, no matter which $n-1$ rows we choose, the determinant of the corresponding submatrix remains the same (upto sign). □

8.3 Adjacency Matrix

Definition 8.3.1

*For a graph G with verices $V(G) = [n]$ and edges $E(G) = \{e_1, \ldots, e_m\}$, the **adjacency matrix** of G is the $n \times n$ matrix $A = (a_{ij})$ defined as follows:*

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

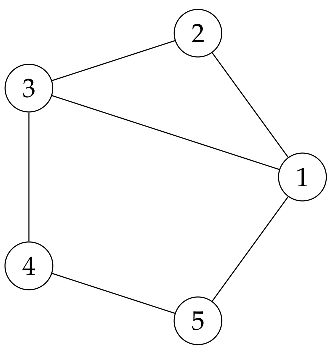

Figure 8.2: Graph G_2

Example 8.3.1

Consider the graph G_2 as undirected version of G_1. It has the adjacency matrix $A(G_2)$ as follows:

$$A(G_2) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Theorem 8.3.1

Let G be a connected graph with vertices $[n]$ and let A be the adjacency matrix of G. The $(i,j)^{th}$ entry a_{ij}^k of A^k counts the number of k-length walks with starting and end vertices i and j respectively.

Proof. By induction on k, the result is evident for $k = 1$. Assuming it holds for $k = m$, consider $A^{m+1} = A^m A$. By induction hypothesis, (i,j)-th entry of A^m counts walks of length m between vertices i and j. Now, the number of walks of length $m+1$ between i and j equals the walks of length m from i to each vertex k adjacent to j. This is expressed as

$$\sum_{k\sim j} a_{ik}^m = \sum_{k=1}^{n} a_{ik}^m a_{kj} = a_{ij}^{m+1}$$

which is precisely the $(i,j)^{\text{th}}$ entry of $A^{m+1} = A^m A$. Hence, the result follows. □

Theorem 8.3.2

Let G be a connected graph with vertices $[n]$ and let A be the adjacency matrix of G. If i, j are vertices of G with $d(i,j) = m$, then the matrices $I, A, \ldots, A^m$ are linearly independent.

Proof. We may assume $i \neq j$. There is no (ij)-path of length less than m. Thus, the (i,j)-element of $I, A, \ldots, A^{m-1}$ is zero, whereas the (i,j)-element of A^m is nonzero. Hence, the result follows. □

Eigenalues of some graphs

Complete grpah, K_n. $\{n-1, \underbrace{-1, \ldots, -1}_{n-1}\}$

Since every vertex of K_n is adjacent to every other vertex, the adjacency matrix consists of all ones except the diagonal entries, which are zero. Thus $A(K_n) = J_n - I_n$, where J_n is the matrix of all ones and I_n is the identity matrix of order n.
Recall that $P_n = \frac{J_n}{n}$ is the projection matrix onto the subspace spanned by the all ones vector. And note that the identity matrix I_n can be decomposed as the direct sum of P_n and $P_n^\perp$, which is the projection matrix of it's complement subspace, i.e. $I_n = P_n + P_n^\perp$. Thus

$$A(K_n) = J_n - I_n$$

$$= nP_n - (P_n + P_n^{\perp})$$
$$= (n-1)P_n + (-1)P_n^{\perp}$$

which is obviously the spectral decomposition of $A(K_n)$. So the eigenvalues of $A(K_n)$ are $n-1$ with multiplicity $\operatorname{rank} P_n = 1$ and -1 with multiplicity $n - \operatorname{rank} P_n = n-1$.

Complete Bipartite grpah, $K_{p,q}$. $\{\sqrt{pq}, -\sqrt{pq}, \underbrace{0,\ldots,0}_{p+q-2}\}$.

Note that,

$$A(K_{p,q}) = \begin{pmatrix} 0 & J_{p,q} \\ J_{q,p} & 0 \end{pmatrix}$$

where $J_{p,q}$ is the $p \times q$ matrix of all ones. Since both $J_{p,q}$ and $J_{q,p}$ have rank 1, $\operatorname{rank} A(K_{p,q}) = 2$, yielding 0 as an eigenvalue with multiplicity $p+q-2$. Let λ be a non-zero eigenvalue of A with an eigenvector $v = \begin{pmatrix} x \\ y \end{pmatrix}$. Now, $A(K_{p,q})v = \lambda v$ implies $J_{p,q}y = \lambda x$ and $J_{q,p}x = \lambda y$. So $qJ_p x = \lambda^2 x$. Since J_p has the eigenvalue p with multiplicity 1 and 0 with multiplicity $p-1$, and $\lambda \neq 0$, we get $\lambda^2 = pq$, which gives $\lambda = \pm\sqrt{pq}$. So the eigenvalues of $A(K_{p,q})$ are 0 with multiplicity $p+q-2$ and $\pm\sqrt{pq}$ each with multiplicity 1.

Cycle graph, C_n. $\left\{2\cos\frac{2\pi k}{n} \mid k = 1,\ldots,n\right\}$.

Note that,

$$A(C_n) = U_n + U_n^T$$

where U_n is the upshift matrix of order n, and $U_n U_n^T = I_n = U_n^T U_n$. Utilizing the eigenvalues of U_n, given by $\left\{\omega^k \mid k = 1,\ldots,n\right\}$ where ω is the n^{th} root of unity, we obtain that the eigenvalues of $A(C_n)$ are $\left\{\omega^k + \omega^{n-k} \mid k = 1,\ldots,n\right\}$.

Definition 8.3.2 ▶ Elementary Subgraph

Let G be a graph with $V(G) = [n]$. A subgraph H of G is called an spanning elementary subgraph if each connected component of H is either an edge or a cycle.

We will denote $c(H)$ and $c_1(H)$ as the number of components of H which are edges and cycles respectively.

Theorem 8.3.3 (Determinant)

Let G be a graph with vertices $[n]$ and adjacency matrix A. Then

$$\det A = \sum_{\substack{H \subseteq G \\ \text{spanning} \\ \text{elementary}}} (-1)^{n - c_1(H) - c(H)} 2^{c(H)}$$

8.4 Laplacian

For a graph G with n vertices and m edges, the **Laplacian matrix** of G is the $n \times m$ matrix L defined as $L = QQ^T$

To be added...