Logistics

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"This will be a reading project not a course."

Goals: i) Learn to communicate mathematics (particularly combinatorics)

ii) In Tao's words, "going from pre-rigourous to rigorous to post-rigourous."

iii) Expect to get acquainted with the process of reading research papers.

Who is this UDGRP right for?

→ Anyone who is at least interested in combinatorics, as it mingles very well with every other branch of mathematics.

And as this is a reading project, we can work out the details about what subfield of combinatorics you can read up on, based on your other mathematical interests.

Prerequisites: At times a lot, but for now none. That's the best part, combinatorics assumes little to no prior knowledge.

Other details: i) For any book, contact me or try Libgen

ii) Feel free to discuss any other research papers that you come across.

iii) Get ready for a lot of random facts

iv) For details visit: Enumerative Combinatorics UDGRP :: Hrishik Koley

Let's Talk

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What are you all interested in?

- Prob → Probabilistic
- NT → Additive
- LA, RA ⇒ fundamental
- Geometry → Geometric

[A Course in Enumeration]

Topological, Analytic, Geometric

- Algebraic Combinatorics [Primer of Analytic Number Theory]
  - Measurable Combinatorics → Measure Theory, Ergodic Theory, Dynamical Systems

What are you all aware of till now?

Generating Functions

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1) Power Series: They are infinite series of the form —

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

[You can think of it as an infinite polynomial]

Example: i) Geometric series: $f(x) = 1 + x + x^2 + \cdots$

$$= \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \; ; \; |x| < 1$$

ii) Taylor series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$,

where $f^{(n)}(c)$ is the n-th derivative of $f(x)$ at $x = n$

2) Generating functions give the closed form for power series.

But what is their purpose?

→ Generating functions store an infinite sequence in a power series, that is expressed in a closed form.

Example: Take the fibonacci sequence, where the n^{th} term is F_n

So, $f(x) = \sum_{n=0}^{\infty} F_n x^n$ encodes the fibonacci sequence in a power series, where

It is also used to give a closed form function for recurrence relations.

3) Too many words, huh. Let's try an example.

$$a_{n+1} = 2a_n + 1, \text{ with } a_0 = 0, \text{ for } n \geq 0$$

Writing down the first few terms of the sequence, we get —

$$0, 1, 3, 7, 15, 31, \ldots$$

Blindly guessing we get that any term $a_n = 2^n - 1$

But that's not what we will do. We will use generating functions.

$$A(x) = \sum_{n \geq 0} a_n x^n$$

$$\sum_{n \geq 0} a_{n+1} x^n = a_1 + a_2 x + a_3 x^2 + \cdots$$

$$= \{(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) - a_0\} / x$$

$n \geq 0$

$$= \{(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) - a_0\}/x$$
$$= \frac{A(x)}{x}$$

$$\sum_{n \geq 0} (2a_n + 1) x^n = 2A(x) + \sum x^n = 2A(x) + \frac{1}{1-x}$$

$$\frac{A(x)}{x} = 2A(x) + \frac{1}{1-x} \Rightarrow A(x) = \frac{x}{(1-x)(1-2x)}$$

4) Methodology:

Given: a recurrence formula that is to be solved by the method of generating functions.

1. Make sure that the set of values of the free variable (say n) for which the given recurrence relation is true, is clearly delineated.
2. Give a name to the generating function that you will look for, and write out that function in terms of the unknown sequence (e.g., call it $A(x)$, and define it to be $\sum_{n \geq 0} a_n x^n$).
3. Multiply both sides of the recurrence by x^n, and sum over all values of n for which the recurrence holds.
4. Express both sides of the resulting equation explicitly in terms of your generating function $A(x)$.
5. Solve the resulting equation for the unknown generating function $A(x)$.
6. If you want an exact formula for the sequence that is defined by the given recurrence relation, then attempt to get such a formula by expanding $A(x)$ into a power series by any method you can think of. In particular, if $A(x)$ is a rational function (quotient of two polynomials), then success will result from expanding in partial fractions and then handling each of the resulting terms separately.

5) A harder example to try is the fibonacci sequence.

$$F_{n+1} = F_n + F_{n-1} \;; \text{ for } n \geq 1,\; F_0 = 0,\; F_1 = 1$$

$$F(x) = \sum_{n \geq 1} F_n x^n$$

$$\sum_{n \geq 1} F_{n+1} x^n = \sum_{n \geq 1} F_n x^n + \sum_{n \geq 1} F_{n-1} x^n$$

$$\Rightarrow \frac{F(x) - x}{x} = F(x) + xF(x)$$

$$\Rightarrow F(x) = \frac{x}{1 - x - x^2}$$

6) What if we have a recurrence relation for a function dependent on two variables n & k.

Ex: $f(n,k) = f(n-1,k) + f(n-1,k-1)$; $f(n,0)=1$

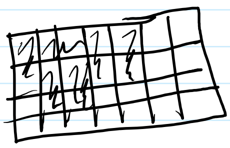
$$B_n(x) = \sum_{k\geq 0} f(n,k)\,x^k$$

$$\sum_{k\geq 1} f(n,k)x^k = \sum_{k\geq 1} f(n-1,k)x^k + x\sum_{k\geq 1} f(n-1,k-1)x^{k-1}$$

$$\Rightarrow B_n(x) - \cancel{1} = \left(B_{n-1}(x) - \cancel{1}\right) + x B_{n-1}(x)$$

$$\Rightarrow B_n(x) = (1+x)B_{n-1}(x) = (1+x)^2 B_{n-2}(x)$$

$$\Rightarrow B_n(x) = (1+x)^n$$



$m \times n$

2×1

$H_n(x) = \sum a_m x^m$

↳ no. of tilings for $m\times n$, where n is fixed

7) You all know the meaning of $\binom{n}{k} \rightarrow$ Given n objects choose k

8) Now what about $\left\{ {n \atop k} \right\} \rightarrow$ Given $\{1,2,\ldots,n\}$, ways to partition into k many classes.

We call this the Stirling number of 2nd kind.

Ex: $\left\{ {4 \atop 2} \right\} = 7$

$\{1\}\{234\}$; $\{2\}\{134\}$; $\{3\}\{124\}$; $\{4\}\{123\}$; $\{12\}\{34\}$; $\{13\}\{24\}$; $\{14\}\{23\}$ $\rightarrow$ 7 (count for yourself)

9) $\left\{ {n \atop k} \right\} = \underbrace{\left\{ {n-1 \atop k-1} \right\}} + k\underbrace{\left\{ {n-1 \atop k} \right\}} \approx 2\times 3 = 6$ $\quad n=4$

OEIS $\rightarrow \left\{ {4 \atop 2} \right\}$

$n-1=3$
$k-1=1$

$\left\{ {3 \atop 1} \right\} = 1$

$\left\{ {4 \atop 2} \right\} = 1+6$

$\{n\}$ $\underbrace{\qquad\qquad\qquad}_{k-1}$

$\left\{ {3 \atop 2} \right\}$

6 ↖ $\{1\}\{2\ 3\}$ ↙ 4

4 ↗ $\{1\}\{23\}$

4 ↗ $\{2\}\{13\}$ ↑ 4

$\{3\}\{12\}$ ↑ 4 ↑ 4

10) The degenerate values:

$$\left\{ {n \atop k} \right\} = 0 \; ; \text{ if } k > n \text{ or } n < 0 \text{ or } k < 0$$

$$\left\{ {n \atop 0} \right\} = 0 \text{ if } n \neq 0 \; , \quad \left\{ {0 \atop 0} \right\} = 1$$

11) Getting to the generating function we have 2 choices for a single variable one.

$$B_k(x) = \sum_n \left\{ {n \atop k} \right\} x^n$$

Use (9), to get –

$$B_k(x) = x B_{k-1}(x) + kx B_k(x)$$

$$\Rightarrow B_k(x) = \frac{x}{1-kx} B_{k-1}(x)$$

$$\boxed{= \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}}$$

$$= \sum_n \left\{ {n \atop k} \right\} x^n$$

$$A_n(y) = \sum_k \left\{ {n \atop k} \right\} y^k$$

$$A_n(y) = \sum_k \left\{ {n-1 \atop k-1} \right\} y^k + \sum_k k \left\{ {n-1 \atop k} \right\} y^k$$

$$= \underline{y A_{n-1}(y)} + \underline{\left(y \frac{d}{dy}\right) A_{n-1}(y)}$$

$$= y(1 + D_y) A_{n-1}(y)$$

$$\Rightarrow A_n(y) = (y + y D_y)^n 1$$

What does this mean?

$$A_1(y) = (y + y D_y) A_0(y) = (y + y D_y) 1 = y$$

$$A_2(y) = (y + y D_y) A_1(y) = (y + y D_y) y = y^2 + y \times 1$$

$$= y^2 + y$$

$$A_3(y) = (y + y D_y) A_2(y) = (y + y D_y)(y^2 + y)$$

$$= y^3 + y^2 + 2y^2 + y$$

$$= y^3 + y^2 + 2y^2 + y$$

$$= \underbrace{y^3 + 3y^2 + y}$$

generatingfunctionology — Herbert S. Wilf

Revisiting Linear Algebra

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1) Definition of vector spaces

2) Determinants of matrices

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 2 & 3 & \cdots & n & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad \pi = \underbrace{(1\ 2)}$$

$\text{sign}(\pi) = +1$ if it can be expressed as prod of even many transposition

-1 " " " " " as prod of odd many transpo.

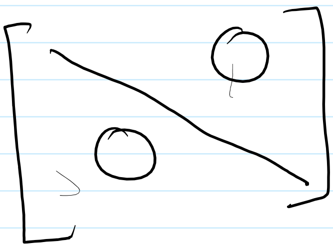
$$\underline{\det(A)} = \sum_{\pi \in S_n} \text{sign}(\pi)\, a_{1,\pi(1)}\, a_{2,\pi(2)} \cdots a_{n,\pi(n)} \ ; \ A_{n\times n}$$

$$a_{11} a_{22} - a_{12} a_{21} = \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right)$$

id , $\pi = (1,2)$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

3) Smith-normal form of matrices



$$\begin{cases} a_{ij} = \begin{cases} 0 & i \neq j \\ \neq 0 & i = j \end{cases} \\ \underbrace{d_{ii} \mid d_{(i+1)(i+1)},} \end{cases}$$

4) Eigenvalues & Eigenvectors (I'll come back if time permits, or you can read by yourself)

Group Theory

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1) Definition of group

2) Subgroups

3) Cyclic groups

4) Symmetric groups

5) Dihedral groups

r, r^2, r^3, r^4

$s = s_x, \underbrace{s_y, s_{ac}, s_{bd}}$

$\{r, r^2, r^3, e, s, sr, sr^2, sr^3\}$, A B C D

$n \to$ sides

$|D_n| = 2n$

$r^3 sr$ $\quad sr^4 = s$ $\to$ DCBA

ABCD

DABC

D A

B C

BCAD

$\to$ DBCA

6) Cosets

G, $H \subseteq G$ $\quad g \in G$

Left Coset: $gH = \{gh \mid h \in H\}$

Right Coset: $Hg = \{hg \mid h \in H\}$

Properties: i) Left cosets of H partition the group G. Each element of G belongs to exactly one coset

ii) All left cosets of H have same size

iii) Correspond to equiv. classes under the relation $\sim$ iff $g_1 g_2^{-1} \in H$

i) $g \in G \Rightarrow g \in gH$ as $e \in H$

$x \in g_1 H \cap g_2 H$

$x = g_1 h_1 = g_2 h_2$

$\Rightarrow g_1 = g_2 h_2 h_1^{-1}$ $\quad g_1 \in g_2 H$

$$\left.\begin{array}{l} g_1 H \subseteq g_2 H \\ g_2 H \subseteq g_1 H \end{array}\right\} g_1 H = g_2 H$$

ii) $\phi : H \to gH \qquad \phi(h) = gh$

ii) $\phi : H \to gH \qquad \phi(h) = gh$

$$\phi(h_1) = \phi(h_2) \Rightarrow gh_1 = gh_2$$
$$\Rightarrow h_1 = h_2 \checkmark$$

$x \in gH \quad \Rightarrow x = gh$ for some $h \in H$

surjectivity $\checkmark$

7) Lagrange's theorem: The no. of distinct cosets of H ($[G:H]$ called the index of H in G) —

$$|G| = |H| \cdot [G:H] \to |H| \,\big|\, |G|$$

Dummitt-Foote

8) Definition of homomorphisms & isomorphisms

9) Quotient group

$$\underbrace{Z_n = Z/nZ}_{} = \{0, 1, ---, n-1\}$$

10) Direct products

$\underbrace{Z_m \times Z_n}_{} \qquad m = 2, n = 3$

Example: $Z_2, Z_3 \qquad Z_2 = \{0,1\},\ Z_3 = \{0,1,2\}$

$$Z_2 \times Z_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}$$

Z_n

Conway-Lagarias $\to$ Honeycomb Tilings

Asymptotics

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"Asymptotics are the calculus of approximations"

1) Big O notation:

Given $C>0$ and $n_0 \geq 0$, $f(n) \in O(g(n))$ if –

$$|f(n)| \leq C \cdot |g(n)| \quad \forall \; n \geq n_0$$

Example: $\underline{3n^2+7n+3 \in O(n^2)}$

$n^2 \in O(n^2)$

$n^2 \in o(n^2)$

2) Small o notation:

Given $C>0$ and $n_0 \geq 0$, $f(n) \in o(g(n))$ if –

$$|f(n)| < C \cdot |g(n)| \quad \forall \; n \geq n_0$$

Example: $n \in o(n^2)$ as $\frac{n}{n^2} = \frac{1}{n} \to 0$ for Large n.

3) Theta (θ) notation:

Given $C_1, C_2 > 0$ and $n_0 \geq 0$, $f(n) \in \theta(g(n))$ if –

$$C_1 \cdot |g(n)| \leq |f(n)| \leq C_2 \cdot |g(n)| \quad \forall \; n \geq n_0$$

Example: $3n^2+5n+2 \in \theta(n^2)$

4) Big Ω notation:

Given $C>0$ and $n_0 \geq 0$, $f(n) \in \Omega(g(n))$ if –

$$|f(n)| \geq C \cdot |g(n)| \quad \forall \; n \geq n_0$$

Example: $3n^2+7n+3 \in \Omega(n^2)$

5) Small ω notation:

Given $C>0$ and $n_0 \geq 0$, $f(n) \in \omega(g(n))$ if –

$$|f(n)| > C \cdot |g(n)| \quad \forall \; n \geq n_0$$

Example: $n^2 \in \omega(n)$ as $\frac{n}{n^2} = \frac{1}{n} \to 0$ for Large n.

6) Abuse of notation: $f(n) \in O(g(n))$

$$\Rightarrow f(n) = O(g(n))$$

7) Properties: $O(n^3 + \underbrace{30n^2+67n+3578}) = O(n^3)$

7) Properties: $O(n^3 + 30n^2 + 67n + 3578) = O(\underline{n^3})$

We ignore lower order terms as after a certain threshold the lower order terms grow at a much slower rate than the highest order term.

Graphs, graphs everywhere

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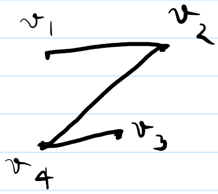
1) Definition

$G = (V, E)$



$$\begin{cases} \underline{\underline{V}} = \{v_1, v_2, v_3, v_4\} \\ \underline{\underline{E}} = \{(v_1, v_2), (v_2, v_4), (v_3, v_4)\} \end{cases}$$

Some Probability

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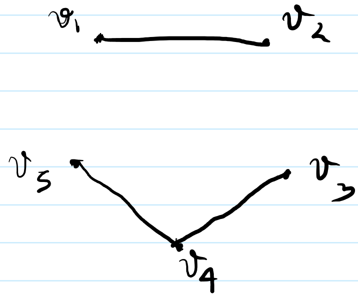
1) Why Probability? How do we introduce randomness?

Probabilistic Method $\longrightarrow$ Combinatorics + Prob

Random Graphs $\longrightarrow$ Probability + Graph Theory

$G_{n,p}$



$n = 5$

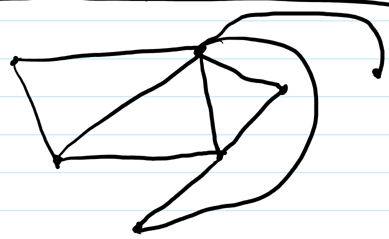
$p = \frac{1}{2}$

${}^5C_2$

Connectivity

$f(n) \leqslant p$

Rich gets richer phenomenon



rich vertex = vertex with high degree

$\mathbb{Z}_n \qquad n \in \mathbb{Z}_+$

finite abelian group G, $n \in \mathbb{Z}_+$

smallest k s.t. every sequence of elements of G of size k contains n terms that sum to 0.

(1961) Erdős-Ginzburg-Ziv:

$\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ $\qquad n \qquad \mathbb{Z}_n \qquad n$

$a_1, a_2, \ldots, a_k$

$k = 2n-1$

$a_{i(1)} + \cdots + a_{i(n)} \equiv 0 \pmod{n}$

Cauchy-Davenport theorem: prime p, $\mathbb{Z}/p\mathbb{Z}$

$A, B \subseteq \mathbb{Z}_p$

$$|A+B| \geq \min\{p, |A|+|B|-1\}$$

$$A+B = \{a+b \pmod{p} \mid a \in A, b \in B\}$$

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$$\underbrace{M(m,n) = 4^{\lfloor m/2 \rfloor \lfloor n/2 \rfloor} \prod_{k=1}^{\lfloor m/2 \rfloor} \prod_{l=1}^{\lfloor n/2 \rfloor} \left(\cos^2 \frac{k\pi}{m+1} + \cos^2 \frac{l\pi}{n+1} \right)}_{m \times n}$$

$m \uparrow \; \uparrow n$

$\underline{m \times n}$ $\underline{\underline{2 \times 1}}$

Dyck paths, Catalan numbers

A Course in Enumeration
↓
Catalan Connection (7^{th} chapter)

Anuvab, Arkaprovo → Catalan
Aayusman → Planar maps (especially graph theory)
Anshuman → Generating Functions
Priyankar → Gen-Func, Partition
Subhojit → Random Graphs (Focus Graph Theory)
Shankha → Additive Combi + Random Graphs