Cauchy-Davenport Theorem

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Theorem: Given some prime p, and non-empty subsets A, B of $\mathbb{Z}_p$ we have

$$|A+B| \geq \min(p, |A|+|B|-1)$$

$$|A+B| = \{a+b \pmod p \mid a \in A, b \in B\}$$

Proof: **Base Case:** $|B| = 1$

$$B = \{b\}$$

$$\text{L.H.S.} = |A+B| = |A|$$

$$|A|+|B|-1 = |A|+1-1 = |A| = \text{R.H.S.}$$

$$|A+B| \geq \min(p, |A|+|B|-1)$$

Induction hypothesis: $|B'| = k$, our statement holds true

Inductive Step: $|A| < p$, $|B| \geq 2$

Case I: $A \cap B$ is a non-empty proper subset of B.

$$A' = A \cup B, \quad B' = A \cap B$$

$$|B'| < |B|$$

$$A'+B' \subseteq A+B \qquad |A'| + |B'| = |A| + |B| \checkmark$$

$$|A+B| \geq |A'+B'| \geq \min(p, |A'|+|B'|-1)$$

$$\Rightarrow \underbrace{|A+B| \geq \min(p, |A|+|B|-1)}$$

Case II: $A \cap B$ is not a non-empty, proper subset of B

i) $A \cap B = \phi$

ii) $A \cap B = B$

$$\underbrace{\exists c \in \mathbb{Z}_p \Rightarrow B \cap (A+c) \text{ is a non-empty proper subset of } B}$$

proper subset of B

$$|B + (A+c)| \geq \min(p, |A+c| + |B| - 1)$$

$(A+c) \rightarrow$ translation/shift of A in $\mathbb{Z}_p$ by c

$$\Rightarrow |B+A| \geq \min(p, |A| + |B| - 1)$$

$$\Rightarrow |A+B| \geq \min(p, |A| + |B| - 1)$$

Planarity

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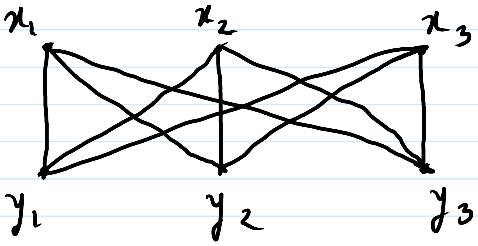
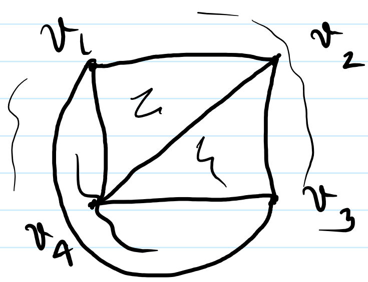


When we have a configuration of a graph G such that no pair of edges intersect then we call such a graph planar.

Theorem (Euler): $V - E + F = 2$

$4 - 6 + 4 = 2$



Polytopes

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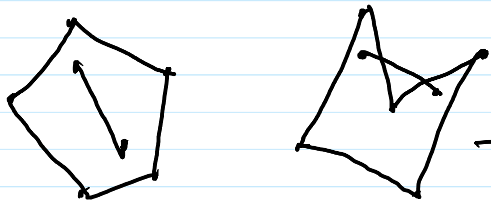
Polygons: P is a convex polygon if for $^{\text{points}}$ $x, y \in P$, line
(Convex) segment xy lies inside P



→ not a convex polygon

3D (Polyhedrons)

v	e	f
8	12	6

Theorem (Euler):

$V - E + F = 2$
(polyhedron)

Theorem (Steinitz): 3-polytope $\Longleftrightarrow$ $\underbrace{\begin{array}{l} v - e + f = 2 \\ v \leq 2f - 4 \\ f \leq 2v - 4 \end{array}}$

5 regular polyhedrons (3-polytopes)

↳ i) each face is a regular polygon of same length (l)

ii) each vertex has same no. of faces converging on it (k)

$\underbrace{kn = 2e = lf}$

$v - e + f = 2$

$\Rightarrow e\left(\frac{2}{k} - 1 + \frac{2}{l}\right) = 2 \Rightarrow \left(\frac{2}{k} + \frac{2}{l}\right) > 1$

$\Rightarrow 2k + 2l > kl$

$(k-2)(l-2) < 4$

$\underbrace{5 \geq k, l \geq 2}$ $\qquad$ $\underline{\underline{5}}$

→ # 30 faces

4D

Eqv. of Euler's theorem: $V - E + F_2 - F_3 = 0$

$F_3 \to$ # 3D faces

$F_2 \to$ # 2D faces

Steinitz theorem in 3D $\not\Rightarrow$ 4D

6 regular 4 polytope (polychora) $\to$ 4 simplex, 4 cube, 4 cross polytope, 24 cell, 120 cell, 600 cell.

Definition: $p_1, \dots, p_n$ in $\mathbb{R}^d$

polytope P = " convex hull of $p_1, \dots, p_n$ "

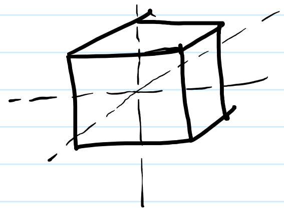
$\hookrightarrow$ minimal convex sets that contains $\{p_1, \dots, p_n\}$;

or set of all convex combinations of points in $\{p_1, \dots, p_n\}$

$$= \left\{ x \in \mathbb{R}^d \;\middle|\; x = \lambda_1 p_1 + \dots + \lambda_n p_n;\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i \geq 0 \right\}$$

Convex Polytopes $\to$ Geometric Combinatorics

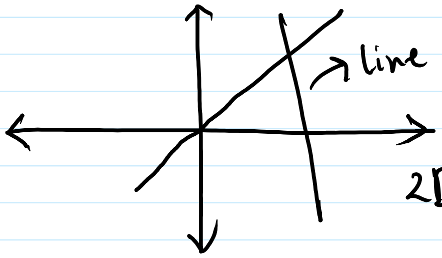
Hyperplane arrangements $\to$ Topological Combinatorics



Lectures on Polytopes — Günter M. Ziegler

What are hyperplanes?

→ A hyperplane is a subspace of one dimension less than the ambient.



line → hyperplane in $\mathbb{R}^2$

2D plane → hyperplane in $\mathbb{R}^3$

<u>Linear hyperplane</u>: i) always passes through the origin

ii) $a_1x_1 + \cdots + a_nx_n = 0$

$a_1, \ldots, a_n$ are constants & not all zero

Ex: $\mathbb{R}^3$, $2x+y-z=0$ → hyperplane

<u>Affine hyperplane</u>: i) translation of a linear hyperplane

ii) $a_1x_1 + \cdots + a_nx_n = b$, $b \neq 0$

Ex: $\mathbb{R}^3$, $2x+y-z=5$ → hyperplane

What is a hyperplane arrangement?

→ $\mathcal{A} = \{H_1, H_2, \ldots, H_m\}$ → finite set of linear/affine hyperplanes in $\mathbb{R}^d$

- $\mathcal{A}$ is central if $\bigcap_{H\in\mathcal{A}} H \neq \phi$
- $\underline{\underline{\mathcal{A}}}$ is essential if the normal vectors on the hyperplanes linearly span $\mathbb{R}^d$

$\underline{\underline{X}} = \text{span}\{H^\perp \mid H \in \mathcal{A}\}$ [Check rank $\mathcal{A}$ $= \dim X$]

$\text{ess}\,\mathcal{A} = \{H \cap X \mid H \in \mathcal{A}\}$

↳ hyperplane arrangement inside $X \cong \mathbb{R}^{\text{rank}\,\mathcal{A}}$

<u>Region of $\mathcal{A}$ $(r(\mathcal{A}))$</u> : connected components of $\mathbb{R}^d - \bigcup H$

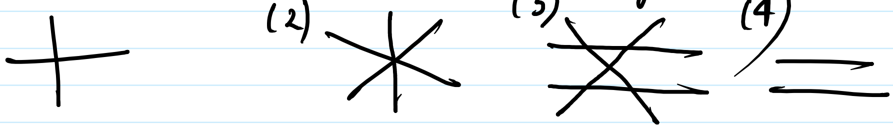
$\mathbb{R}^d - \bigcup_{H \in \mathcal{A}} H$.

$r(\mathcal{A})$ = # regions of $\mathcal{A}$

Region R of $\mathcal{A}$ → relatively bounded if $R \cap X$ is bounded

$b(\mathcal{A})$ = # relatively bounded regions of $\mathcal{A}$

$\mathcal{A}$:



	(1)	(2)	(3)	(4)
central:	✓	✓	×	×
essential:	✓	✓	✓	×
rank $\mathcal{A}$:	2	2	2	1
$r(\mathcal{A})$:	4	6	10	3
$b(\mathcal{A})$:	0	0	2	1

Lecture notes on Hyperplane Arrangements
— Richard P. Stanley

✓ Simplicial Complexes →

Noga Alon & Moshe Dubiner

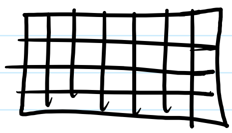
(Erdös – Gizburg, Ziv)
$a_1, \ldots, a_{6n-5}$ (not necessarily distinct) members of the group $\mathbb{Z}_n \oplus \mathbb{Z}_n$, there is a set $I \subset \{1, \ldots, 6n-5\}$ of cardinality $|I| = n$ so that $\sum_{i \in I} a_i = 0$ (in $\mathbb{Z}_n \oplus \mathbb{Z}_n$)

$a_1, \ldots, a_{2n-1}$, $\mathbb{Z}_n$

Prob. Combi.

The Probabilistic Method: Noga Alon & Joel Spencer

Lovasz Local Lemma → Spring 2024 (MC, IB)



$m \times n$

2×1 – dominoes

$k \times 1$ $k = 3$

$k < m < 2k$

$m \times n$ $k \times 1$