Sperner's lemma and it's interesting applications

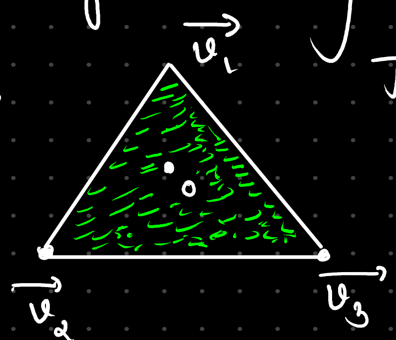
- Sperner's lemma is a gem in combinatorics, first proposed by Emmanuel Sperner.
- Sperner's lemma was originally devised to give a simple proof of perhaps one of the most elementary, yet profound result in early Algebraic Topology — "Brouwer's Fixed point Theorem"
- Sperner's lemma being a gem, also found its way into the theory of Nash Equilibrium and some fair division problem.
- Naively, Sperner's lemma states that for a coloring of a "simplicial subdivision" of a "simplex" satisfying certain boundary conditions, there must be atleast one "rainbow cell", recieving all possible colours. — Let us discuss these terms explicitly.

Some notations and definitions:

- $\vec{v} = (v_1, \ldots, v_k)$ is a k-dimensional vector. e_i's represent the canonical vectors.

- By $\text{conv}(\vec{v}_1, \ldots, \vec{v}_k)$, we represent the smallest convex set spanned by the vectors v_i, $i \in [k]$

Eg:



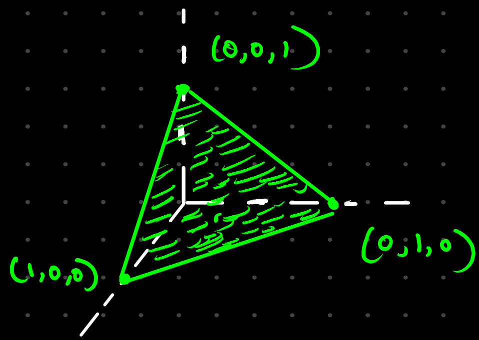
Treat the vectors as position vectors. Notice $\text{conv}(\vec{v}_1, \ldots, \vec{v}_k)$ is indeed the smallest convex set.

Can you observe that $\text{conv}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is a triangle? If not, first think about the convex hull of two vectors first.

⊙ A K simplex is defined by:

$$\Delta_K = \operatorname{conv}(e_1, \dots, e_{K+1}) = \left\{ x = (x_1, \dots, x_{K+1}) \in \mathbb{R}^K : x_i \geq 0 \text{ and } \sum_{i=1}^{K} x_i = 1 \right\}$$

If we want to visulaise Δ_3, it would look like.

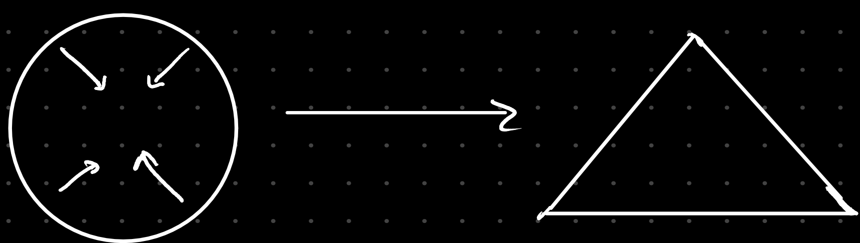


⊙ However from topology pov, any two convex hull spanned by the same no. of vectors is somewhat the same. We call two of them to be homeomorphic. Strictly defining:

If X and Y are two topological spaces, they are homeomorphic if $\exists$ a continuous map $g : X \to Y$ such that g is continuous and bijective, with it's inverse also being continuous.

⊙ Let us understand this Naively:

take a circle and "squish" it to form a triangle:

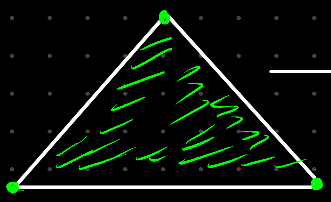


⊙ Also using similar squishing, stretching and inflating techniques, we can deduce any two triangles are homeomorphic

⊙ So from now on any K-simplex, i.e, defined by any $K+1$ vectors will be denoted by Δ_K.

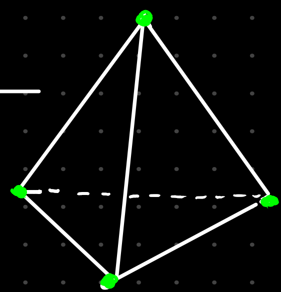
⊙ Given any n-simplex, any face spanned by n of the $n+1$ vertices of S is called a "facet". Observe that a n-simplex has $n+1$ facets.

Eg:



The facets are the three line segments, as facets are of the form $\mathrm{conv}(v_1, v_2)$.

Similarly in the case of tetrahedrons, the facets are the triangles.

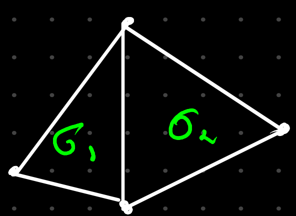


⊙ Let us understand Simplicial Subdivision. Sps a n-simplex Δ_n is given. Then a simplicial subdivision of Δ_n are collection of n-simplices such that:

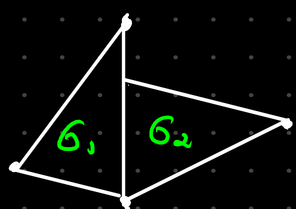
⟶ Union of the simplices gives us the whole simplex

⟶ Let σ_1, σ_2 be two cells in the collection. Then their intersection is either empty, or a full face of a certain dimension shared by σ_1 and σ_2

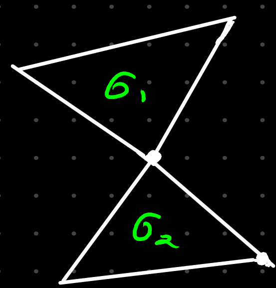
Eg:



This is a valid intersection



This is invalid (A full face is not shared)



This is also valid

Sperner's Lemma:

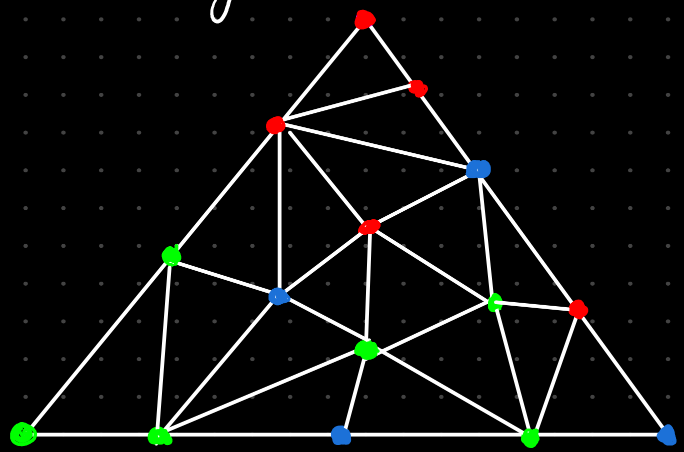
⊙ Let us understand Sperner labelling on a triangle first.

→ The three vertices of T must all have different colours

→ If E is an edge of T, then each vertex on edge E must have the same colours as of the end points of E

→ The interior of T can have any colour.

Consider the labelling:



This forms a valid sperner labelling.

Sperner's Lemma for Triangles: Any Sperner-labelling of a triangle must contain an odd number of elementary triangles having all different labelled colours.

Corollary: Any Sperner-labelled triangulation of a triangle must contain atleast one elementary triangle having all different labeled colors.

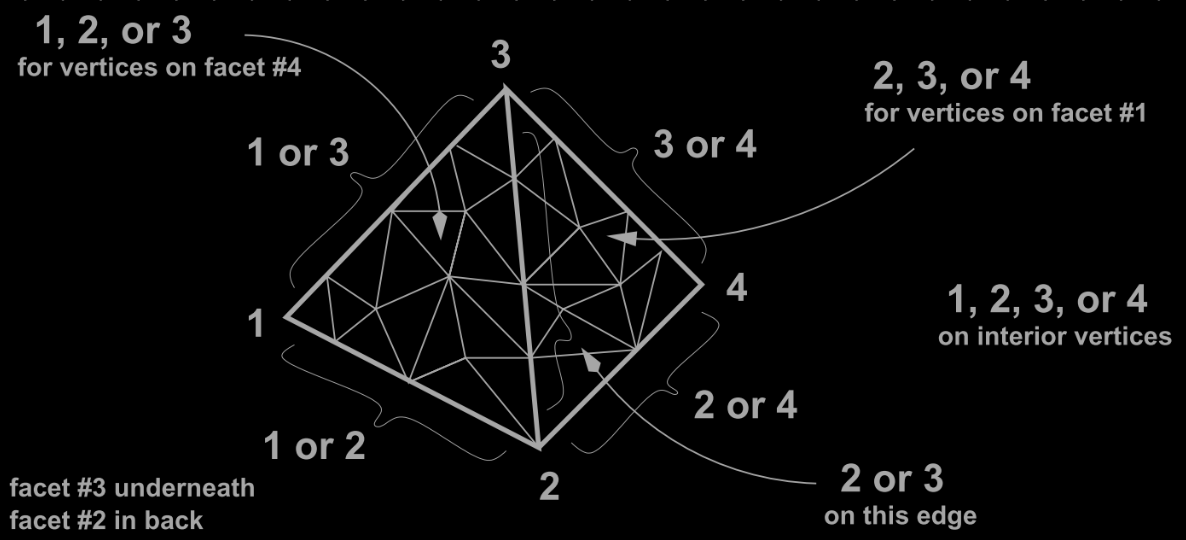
 # Note: In many applications we only use the fact that atleast one such triangle exists.

n-dimensional Sperner Labelling:

Number the facets of S by 1, 2, ..., n+1. Then the labelling follows the following rules:

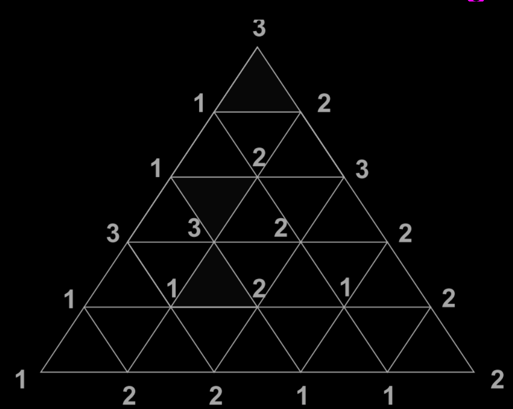
- → Each vertex have a different number
- → Each vertex is labelled by one of the facet numbers in such a way that on the boundary of S, none of the vertices on facet j is labelled j.

For example, consider the subdivision of the tetrahedron:



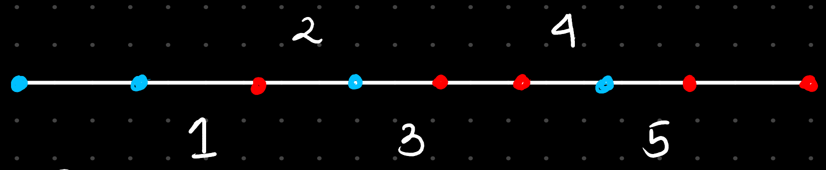
n-dimensional Sperner's Lemma: Any Sperner Labelling of a n-simplex must contain an odd number of fully labelled elementary n-simplices. In particular there must be atleast one.

#Disclaimer: To understand the proof, we look at the running example of the following 2 simplex:



proof: We proceed by induction on dimension n.

$\underline{n=1}$: A simplicially subdivided 1-simplex is just a segmented line segment. The end points are of diff. colors, say 1 and 2. Observe that in moving from point 1 to point 2, the labelling must switch an odd no. of times (O.W the end points have the same colour)

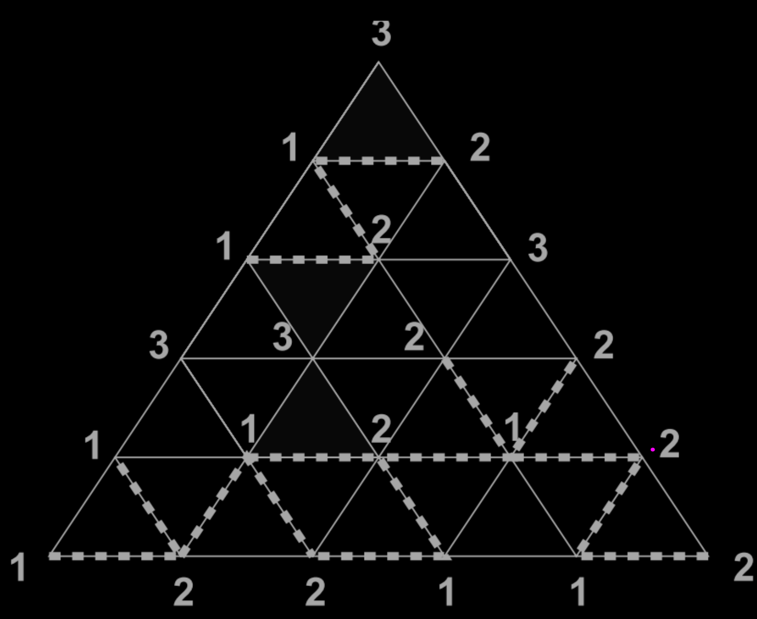


Now Assume that the theorem holds for dimensions up through $(n-1)$. We show that this is true for a simplicially subdivided n-simplex, say S. Let us make this proof interesting:

- Say our n-simplex S is a "house", subdivided into many "rooms".

- A facet of the room is called a "door" if that facet carries first n of the $(n+1)$ labels. In the running example, these are the $(1,2)$ edges.

For the case of $n=3$, the doors are triangles with vertices $(1,2,3)$.



Now the only facet containing the doors is the $(n+1)$st facet. By induction hypothesis, we conclude that the no. of doors on the facet are odd, i.e, there are odd no. of fully labelled $(n-1)$ simplices on the facet. We call them Boundary doors.

Observation: Every room can have atmost 2 doors, and it has exactly 1 door iff the room is fully labelled in S.

(Excercise: Check this)

for the $n=3$ case verify that a tetrahedron with labels $\{1,2,3,3\}$ has two doors.

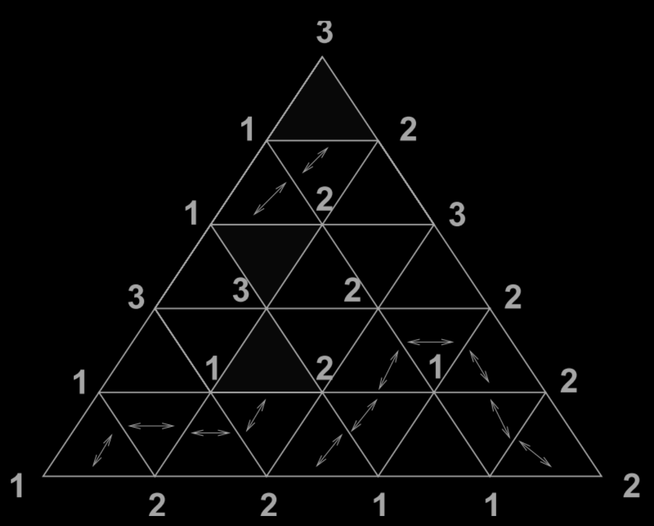
We now use the boundary rooms to locate the fully labelled rooms. We now use the "trap door" argument:

Start at any boundary door, and walk into the adjoining room. Either this room is fully labelled or it has one other door, which we call the trap door that we can walk through. Keep on Repeating this procedure. As each room has atmost two doors, we can't visit a room twice. As the no. of rooms are finite, this procedure must end. There are two possible endings:

(i) We walk into a fully labelled room

(ii) Use a boundary door to exit the building.

As the no. of boundary doors is odd, and the trap door paths (i.e paths involving only trap doors) must pair up an even no. of boundary doors, The no. of boundary doors left over, which leads to fully labelled rooms / Rainbow cells, must be odd.



So if we show that the rainbow cells, which are not accessible via boundary doors, are even, then total no. of rainbow cells are odd. But this is easy: two such rainbow cells are paired up by their own trap door paths. (If this Argument is not clear, try searching for a contradiction)

Thus total number of rainbow cells are odd ▨.

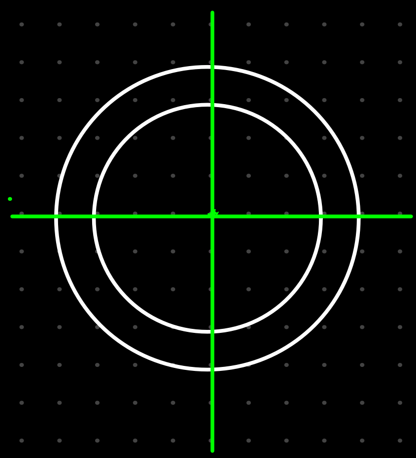
Remarks:

⊙ From the last Argument, Sperner's lemma $(n=2)$ can be Reformulated as:

If we have Sperner labelled triangle, then the no. of rainbow triangles is equal to the number of $(1,2)$ edges on the boundary of the polygon modulo 2.

Brouwer Fixed Point Theorem

$\mathbb{D}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \|x\|_2 \leq 1\}$. This is also called the n-dimensional ball. One might wonder, Why Unit ball? Why not ball of radius 2 (or any $r>0$)? Again the notion of homeomorphism becomes handy. Let us see this for $n=2$ case. Let B_r be the 2-D Ball centred at the origin with radius r



⊙ Consider B_1 and B_2 as shown.

Points on B_R are of the form $re^{i\theta}$ where $0 \leq r \leq R$ and $\theta \in [0, 2\pi)$

Define
$$\varphi : B_1 \longrightarrow B_2$$
$$re^{i\theta} \longmapsto 2re^{i\theta}$$

Exercise:

(i) Check that φ is indeed a homeomorphism.

(ii) Can you extend this to the n dimensional ball?

Summing up:

Every theorems or statements in Topology always come with an invisible assumption: "Upto homeomorphisms"

The same thus goes for Brouwer Fixed point theorem.

Brouwer Fixed Point theorem:

Any continuous map $f: \mathbb{D}^n \longrightarrow \mathbb{D}^n$ has a fixed point.

Before starting the proof, consider $n=1$ case:

$f: [-1,1] \longrightarrow [-1,1]$ has a fixed point. Consider

$g(x) := f(x) - x : [-1,1] \longrightarrow [-1,1]$. Now by IVT, f has a fixed point if g takes both positive and negative values.

Assume (WLOG) that $f(x) > x \; \forall x \in [-1,1] \Rightarrow f(1) > 1 \Rightarrow\Leftarrow$

So f has a fixed point. This was nothing but basic RA.

Let us see the proof for $n=2$ case:

proof: $\mathbb{D}^2$ is a unit disk, homeomorphic to a solid triangle.

Let $\Delta \subseteq \mathbb{R}^3$ with vertices e_1, e_2, e_3. Thus

$$\Delta = \{(a_1, a_2, a_3) \in [0,1]^3 : a_1 + a_2 + a_3 = 1\}$$

Assume that $f: \Delta \longrightarrow \Delta$ has no fixed points.

For each vertex $v \in T$, define colouring of v to be the minimum color i such that $f(v)_i < v_i$, i.e i^{th} coordinate of $f(v) - v$ is negative. (Check that this colouring is well defined!)

Claim: e_1, e_2 and e_3 have different colours.

proof: Seriously?

Observe that vertex on $e_1 e_2$ has $a_3 = 0$, so $f(v) - v$ has non negative third colouring, and hence is colored 1 or 2. Same for the other two coordinates. So $\exists$ a Sperner labelling for any triangulation T_K. By Sperner's lemma (Weak version) $\exists$ a complete triangle Δ_K.

Denote by $\delta(T)$, the "maximal" length of an edge in a triangulation T. By repeated subdivisions, we can construct a sequence of triangulations $T_1, T_2, \ldots$ such that $\delta(T_K) \longrightarrow 0$ as $K \longrightarrow \infty$.

Note that $\{\Delta_K\}$ don't necessarily have to converge. But we know that "Every sequence in a compact space has a convergent subsequence"

So $\exists$ a subsequence $\Delta_{K_1}, \Delta_{K_2}, \ldots$ converging to some point x.

Hence we get a sequence of red, green and blue points, say $\{r_n\}_{n \in \mathbb{N}}$, $\{g_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ converging to x

So $\quad f(r_n)_1 < (r_n)_1 \;;\; f(g_n)_2 < (g_n)_2 \;;\; f(b_n)_3 < (b_n)_3$

Taking limit on both sides, from continuity of f, we have

$$f(x)_1 \leq x_1 \;;\; f(x)_2 \leq x_2 \;;\; f(x)_3 \leq x_3$$

Also we have $x_1 + x_2 + x_3 = f(x_1) + f(x_2) + f(x_3)$

$$\implies f(x) = x \quad \blacksquare$$

Some remarks:

We have used Compactness without properly defining it. Let us see some equivalent definitions:

Let (X,d) be a metric space, and $A \subseteq X$

(i) A is compact iff every open cover has a finite subcover

(ii) A is compact iff it is totally bounded and complete

(iii) A is compact iff every sequence in A has a convergent subsequence

In the above proof, our underlying space was $\mathbb{R}^3$. In general for the Euclidean space $\mathbb{R}^n$, $n \geq 1$, we use Heine-Borel Theorem for detecting Compact Sets:

$A \subseteq \mathbb{R}^n$ is compact iff it is closed and Bounded.

Some problems on compactness:

(i) Is the set $\{(x,y,z) : x^2+y^2-z^2=1\} \subseteq \mathbb{R}^3$ compact?

Is $\{(x,y,z) : x^2+y^2+z^2=1\}$ compact?

(ii) Let (X,d) be a metric space, and $A \subseteq X$. Show that:

$\rightarrow$ If A is compact, then A is closed in X

$\rightarrow$ If X is compact and A is closed, then A is compact

Some More on Brouwer's Fixed Theorem:

- This is a lengthy yet simple proof of BFT.
- There are other proofs, using notions from Alg. Topo like homotopy, Fundamental groups, Deformation Retractions and Homology, which is Beyond our scope.
- BFT is a very elementary yet very important theorem in Alg Topo, being the building block for many complex theorems.

Some Further Applications of Sperner's Lemma:

- Sperner's Lemma is used to prove Monsky's Theorem:

  It is not possible to dissect a square into an odd number of triangles of equal area.
- Sperner's Lemma helps in Solving some "envy free" division problems — division in which a person doesn't envy any other person.

  Some of them are the Cake cutting problem and the Rent partitioning problem.

Some References and Reading Materials:

- https://math.berkeley.edu/~moorxu/misc/equiareal.pdf

This contains the proof of Monsky's Theorem.

- https://ocw.mit.edu/courses/18-304-undergraduate-seminar-in-discrete-mathematics-spring-2015/4ba52e9fdecead1d55c206ef4abb0ed3_MIT18_304S15_project1.pdf

This contains more about Fair Division Problems.

- You can consult books like Munkres or Kumaresan for results in basic point-set Topology.