Kuratowski's Thm

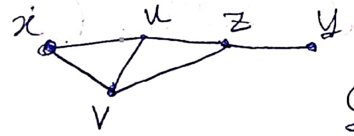
------- : Path

—— : edge

Definitions :

Vertex cut set : It is a set of vertices in graph G s.t. removing those vertices disconnects the graph.

eg :



G

Here, $\{u, v\}$; $\{u, v, z\}$; $\{z\}$ are some vertex cut sets.

K-Connected : A graph G is called K-connected if the size of the minimum vertex cut set is K (or more than K).

eg : In the previous graph G, $\{z\}$ is a vertex cut $\therefore$ G is 1-connected.

$d_m(u, v)$ = no of edges in the path b/w u and v s.t. this path is of minimum lenght (min edges).

eg : In the above graph G,

$d_m(u, y) = 2$; $d_m(x, y) = 3$

Lemmas :

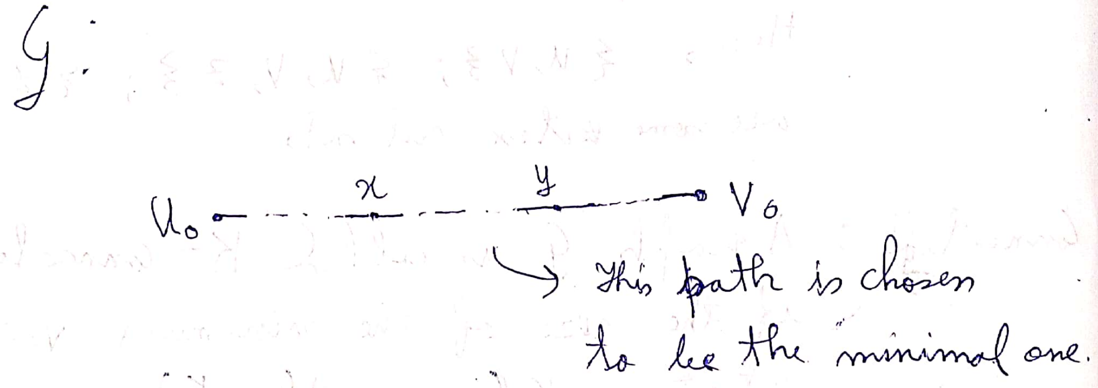
L_1 : A graph G is 2-connected, ~~then~~ iff each vertex pair u, v lie in a cycle. $[u \neq v]$

L_2 : If a graph G is 3-connected and $e \in E[G]$ then, $G - e$ is 2-connected

[$\Rightarrow$]

Proof of L_1 : Choose, $u_0, v_0 \in V[G]$ s.t. u_0, v_0 does not lie in a cycle and $d_m(v_0, u_0)$ is minimized along all such vertex pairs.

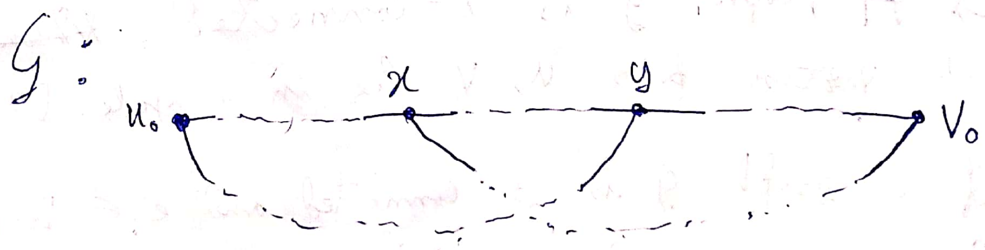
Assume, $d_m(u_0, v_0) \geq 3$, i.e. there are atleast 3 edges in the minimal path. ($\because$ we have 2 vertices atleast)



Clearly, $d_m(u_0, y) \leq d_m(v_0, u_0)$

$\Rightarrow$ u_0, y lie on a cycle or else v_0, u_0 wont be the minimal ~~such~~ vertex pair with that property.

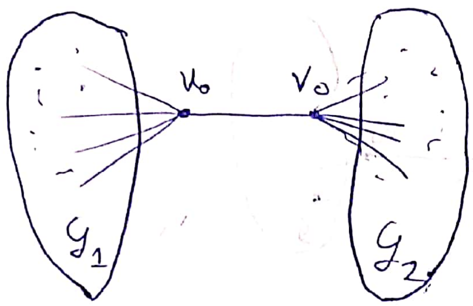
[Note : $d_m(u_0, y) < d_m(v_0, u_0)$ as the path on the minimal path from u_0 to y is already less than $d_m(v_0, u_0)$]



If the path from $u_0 \to y$ and path from $v_0 \to x$ share any vertices say z then, $(u_0 v_0 z u_0)$ is a cycle. If not then, $(u_0 x v_0 y u_0)$ is a cycle. which is a contradiction!

$\Rightarrow$ $d_m(u_0, v_0) = 1$ or 2 [If $d_m(u_0, v_0) = 0$ then $u_0 = v_0$ $[\Rightarrow\Leftarrow]$]

C_1: $d_m(u_0, v_0) = 1$



Clearly $\nexists$ $u \in V[G_1]$, $v \in V[G_2]$ s.t. $\exists$ an edge b/w u and v or else $(u_0 v_0 v u u_0)$ is a cycle.

But if so then, $\{v_0\}$ is a cut vertex set making G, 1-connected. $[\Rightarrow\Leftarrow]$

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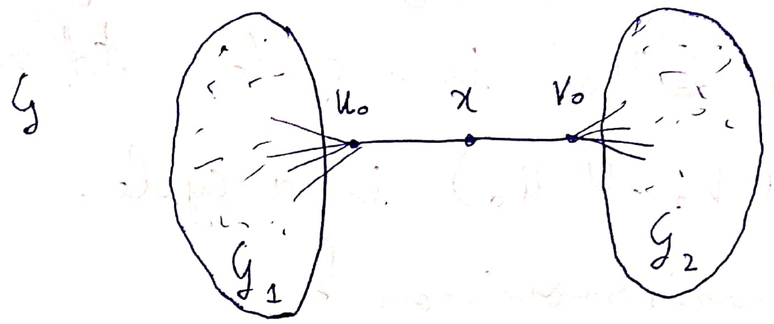
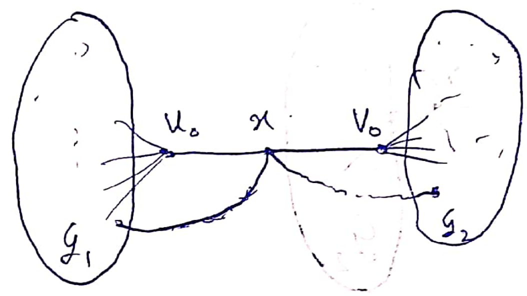
$C_2 : d_m(u_0, v_0) = 2$



The same logic as in C_1 applies here if ~~G~~ both G_1 and G_2 are not connected to x.

$\Rightarrow$ G :



Notice, $\{x\}$ is ~~not be~~ now a cut vertex set if we insist on u_0, v_0 not lying on a cycle.

$\Rightarrow$ G is 1-connected

$[\Rightarrow \Leftarrow]$

$\therefore$ Our initial assumption was wrong meaning if G is 2-connected.

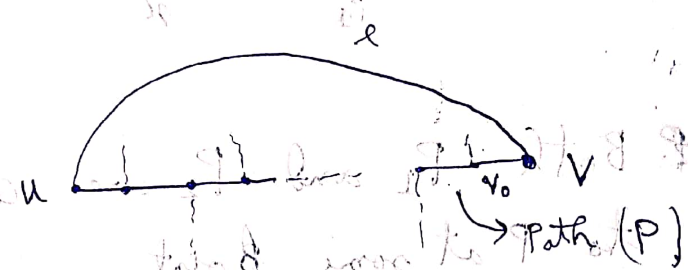
$\forall\, u, v \in V[G]$ $u \neq v$, u, v lie in a cycle.

[$\Leftarrow$]

Now if $\forall\, u, v \in V[\mathcal{G}]$, $u \neq v$; u, v lie in a cycle then, $\{u\}$ or $\{v\}$ is never a vertex cut. $\therefore$ $\mathcal{G}$ is 2-connected.

Proof of L_2 : Let $\mathcal{G}$ be 3-connected and, $e \in E[\mathcal{G}]$ s.t. $u, v \in V[\mathcal{G}]$ are the ends of e. Note : $\exists$ another path from u to v or else $\mathcal{G}$ is a graph with 2 vertices which is not 3-connected. Let P be the minimal of such paths.

$\mathcal{G}$:



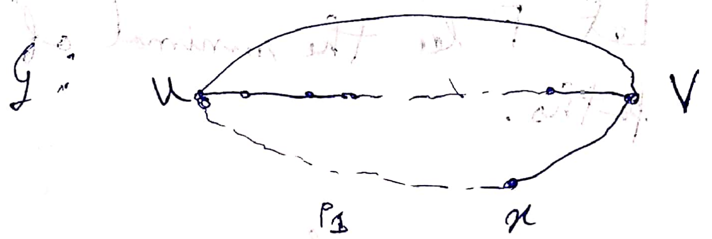
If v does not connect to anything else then, $\{u, v_0\}$ is a cut vertex set. [$\Rightarrow\Leftarrow$] $\therefore$ Say v connects to some x.

If, sps, x was on the path P then clearly $u - x - v$ is a more minimal path than P [$\Rightarrow\Leftarrow$].

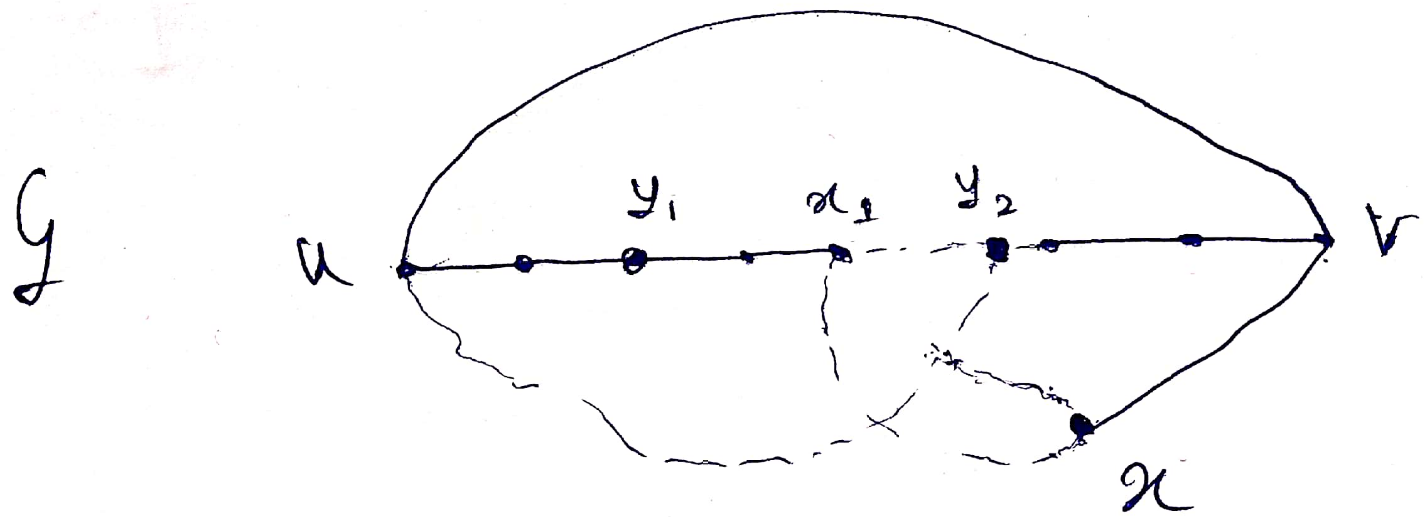
$\therefore$ x lies outside P.

As G is also 2-connected
$\Rightarrow$ x and u lie in a cycle
i.e. $\exists$ 2 disjoint paths from x to u
say P_1 and P_2.

C_1: P_1 or P_2 is disjoint from P.
Then we have a cycle that
u and v lies on and we are done.
(Cycle in $G-e$)



C_2: Both P_1 and P_2 are connected
to P at some point.
Let, P_1 be the path that
lies in P closest to u.
Sps, the ~~closet~~ closest point is y_1
and this started from x_1.
And Let, y_2 be the point
closest to v.



Then, $(u\, y_1\, x_1\, x\, v\, y_2\, u)$ is a cycle

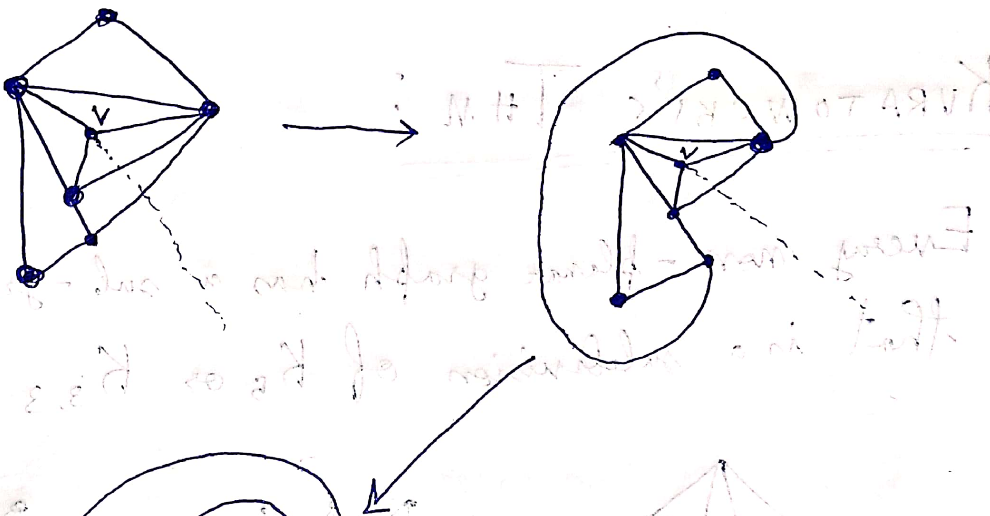
$\therefore$ Again, u, v lie in a cycle in $G - e$.

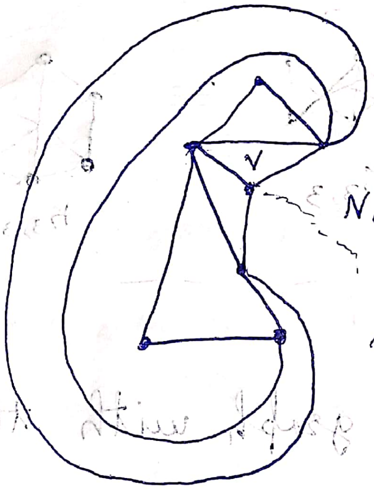
$\Rightarrow$ $G - e$ is 2-connected.

Inversion of a planar embedding :

Given any planar map of a graph G and a vertex V in G we can bring that vertex to the exterior of G while retaining planar and graph structure.

eg :





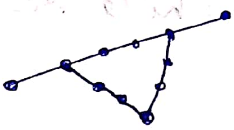
Now V is in the exterior of G while still maintaining planarity.

Pf idea : Draw a line from V to the exterior without cutting any vertex then ~~the~~ to the outermost edge take it to the back.

Note : This can also be done with an edge.

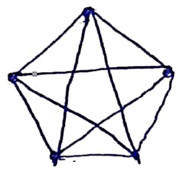
Subdivision of a graph:

A subdivision of a graph is basically adding more vertices to an edge.

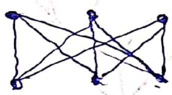
eg:  is a subdivision of, 

KURATOWSKI'S THM:

Every non-planar graph has a sub-graph that is a subdivision of K_5 or $K_{3,3}$

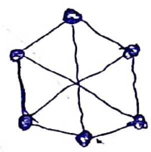


K_5



$K_{3,3}$

$\equiv$



$K_{3,3}$

PROOF: Let, G be the graph with the following properties:

(I) No subgraph of G contain subdivision of K_5 or $K_{3,3}$

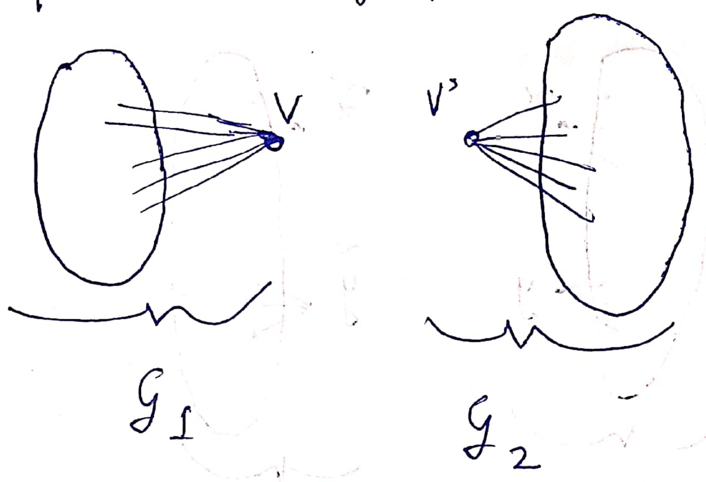
(II) G is non-planar

(III) G is the smallest graph with these properties.

Claim: G is 3-connected.

Pf: Sps, G is 1-connected and not 2-connected.

$\therefore \exists$ V vertex in G s.t. cutting at V seperates the graph.

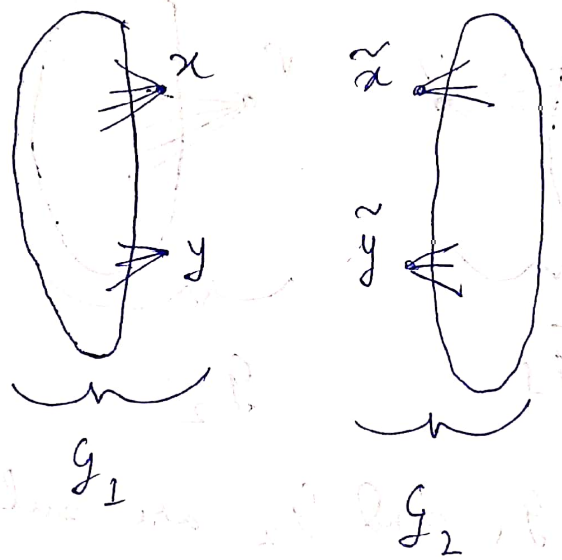


Clearly, G_1 and G_2 are subgraphs of G. $\therefore$ By minimality of G: Both G_1 and G_2 have a planar embedding, or one of them contains K_5 or $K_{3,3}$ [if the later was the case then G would contain K_5 or $K_{3,3}$ $\therefore$ that is not possible]

$\Longrightarrow$ Take the planar embedding of G_1 and G_2 and insert around V and V' then ~~gl~~ join them obtaining a planar embedding of G [$\Longrightarrow\Longleftarrow$] ($\because$ G is ~~not~~ 2-connected)
(at least)

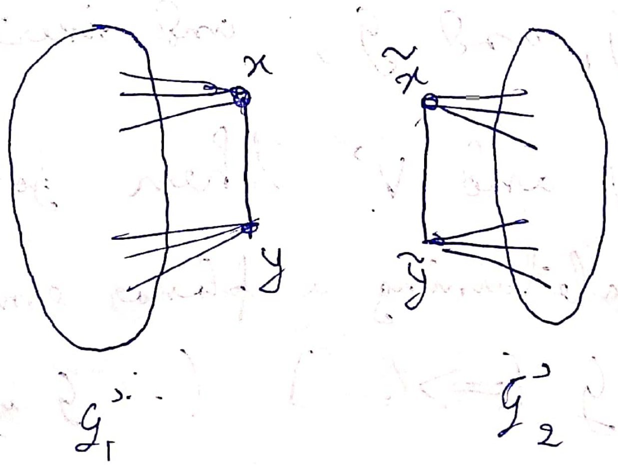
Now, Sps G was 2-connected but not 3-connected.

$\therefore \exists$ 2-vertices in G say x, y s.t. if we cut them it disconnect's the graph.



By same logic, G_1 and G_2 are planar. The same inversion trick will not work on two vertices as whenever we try to invert, we may not have both outside G_i.

Construction: Draw an edge b/w x any y in both G_1 and G_2 and call the new graphs G_1', G_2'.



Now if both G_1' and G_2' are planar then we can invert around the edge like we inverted around a vertex and bring it out then connect x to $\tilde{x}$ and y to $\tilde{y}$ then ~~so~~ remove the edge ~~obtaining~~ obtaining ~~an~~ a planar embedding of G.

$[\Rightarrow \Leftarrow]$

(Wlog sps G_1' is not planar)

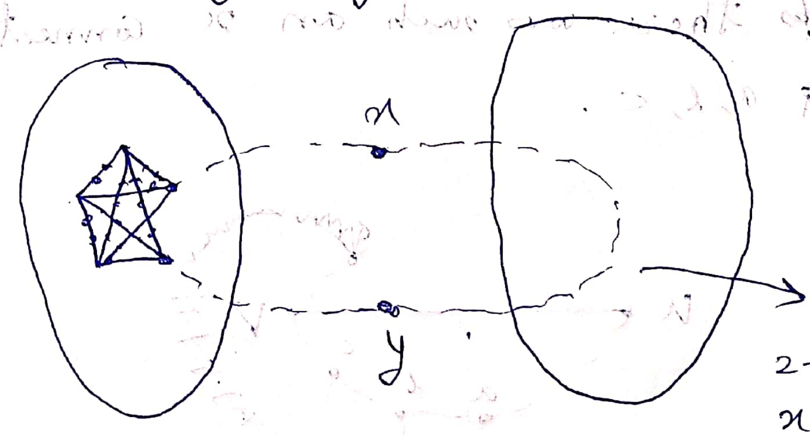
$\therefore$ Atleast one of G_1', G_2' is non-planar.

Due to minimality of G, G_1' must contain subdivisions of K_5 or $K_{3,3}$. (or else G_1' would have been the minimal G).

as, $G_1' - xy = G_1$ is planar it does not contain subdivision of K_5 or $K_{3,3}$. $\therefore$ the edge xy is in this subdivition of K_5 or $K_{3,3}$.

Now join G_1 and G_2' and get G' then remove the xy edge.

G



as G is 2-connected xy lie in a cycle.

$\Rightarrow \exists$ Subdivision of K_5 or $K_{3,3}$ within G.

$[\Rightarrow \Leftarrow]$

$\Rightarrow G$ is 3-connected (at least)

Now we know :

(I) G is 3-connected

(II) G is non-planar.

(III) G has no subdivision of K_5 or $K_{3,3}$

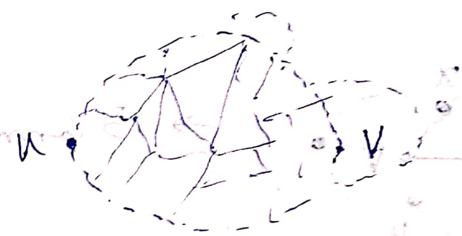
(IV) G is the minimal graph of the following property.

Let u, v be two ~~of~~ adjacent vertices. Due to minimality of G ; $G - uv = G'$ must be planar. [Otherwise if $\exists$ subdivisions of K_5 or $K_{3,3}$ in G' then they are also in G [$\Rightarrow\Leftarrow$]]

Now by L_2 : $G - uv = G'$ must be 2-connected [as G was 3-connected]

Now consider ~~a~~ some planar embedding of G' and in it find the cycle containing u, v (This exists via L_1) s.t. this cycle contains the maximum amount of region in it.

G'



If, $\exists$ an edge on a cycle connecting back to the cycle outside the interior structure of this planar embedding, then this edge must connect one path to the other.
As otherwise,

G'

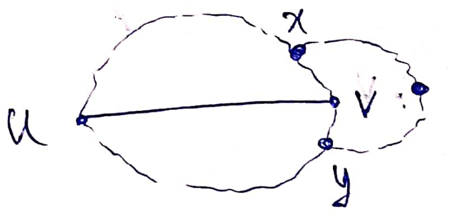
Now we have that $u\,x\,y\,v\,u$ is the ~~mi~~ cycle with maximum regions inside it.

Notice, an exterior vertex cannot connect to G in ≥ 3 places or else by PHP two edges connect back to the same path from $u \to v$ on this cycle.

[Also if it connected to one place then G' would be 1-connected $[\Rightarrow\Leftarrow]$.]

For none ofs this exterior structure exists in G' and therefore in G.

G :



$\rightarrow$ 3 connected

Notice here, $\{x, y\}$ is a vertex cut set of G if xy is not a direct edge.

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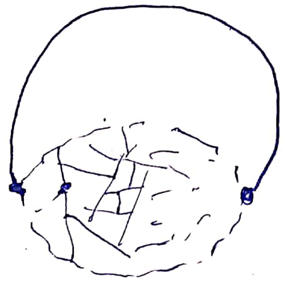
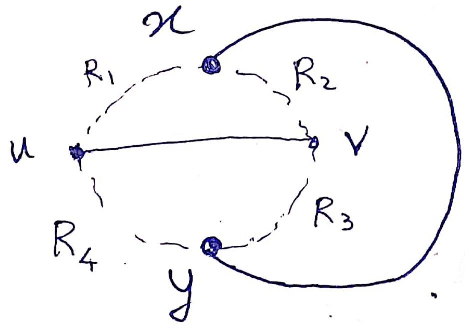
∴ If this structure exists then it must be a direct edge ~~for~~ from one path to another.

Sps this structure did not exist in G' and we know G' is planar but then this means G is also planar (absurd!)



→ add back the edge without any ~~obstruction~~ obstruction.

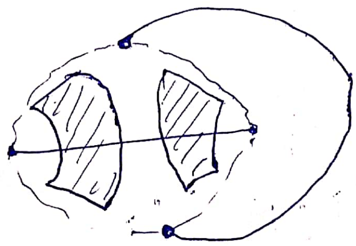
∴ G :



→ This is non planar.

(R_i's are the paths ~~regions~~ on the cycle)

Note: If the internal structure is like this then G is planar,

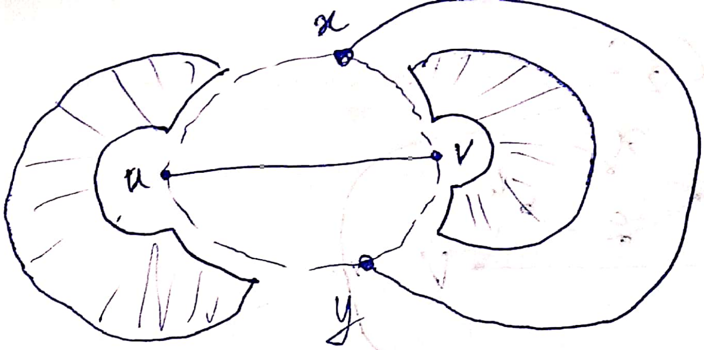


[i.e if ∃ no connection from R_1 to R_3 or R_2 to R_4]

This is because G' is planar.

So ∃ an planar embedding of G'.

Planar embedding of G:

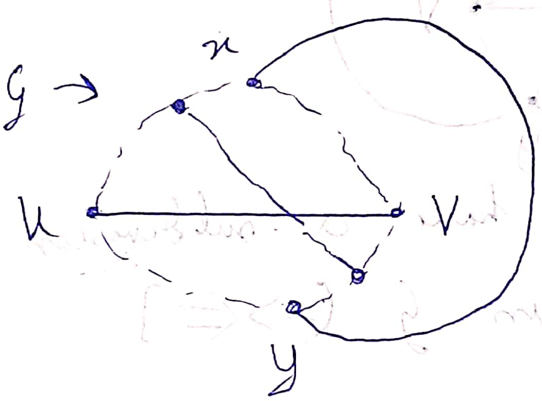


$[\Longrightarrow \Longleftarrow]$

$\therefore \exists$ at least one connection from R_1 to R_3 or R_2 to R_4

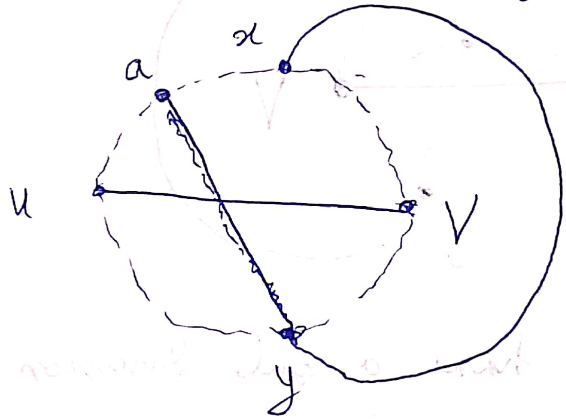
Wlog sps $\exists$ an connection b/w R_1 to R_3

C_1: $G \to$



$\to$ this has a subgraph which is a subdivision of $K_{3,3}$.

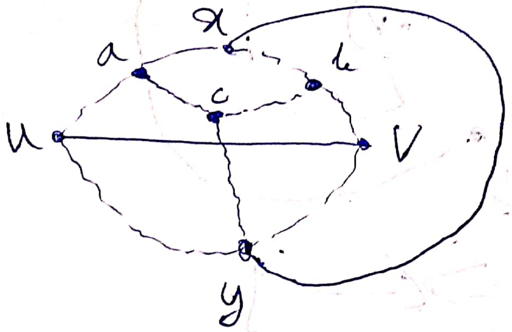
C_2:



But if this was the only structure then G has a planar embedding.

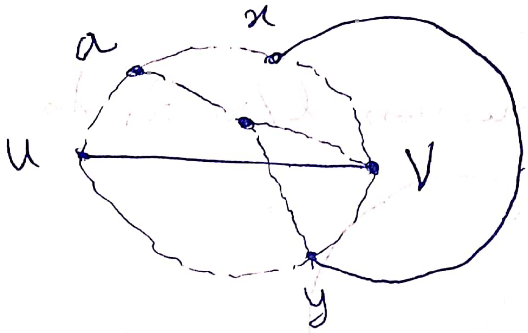
Subcases of C_2:

(I)



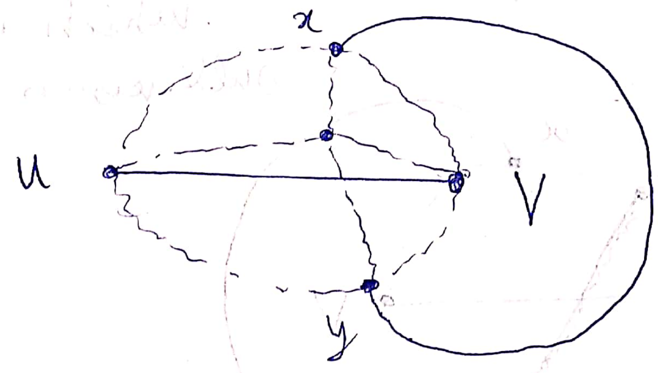
again here we have a sub Division of $K_{3,3}$ [$\Rightarrow\Leftarrow$]

(II)



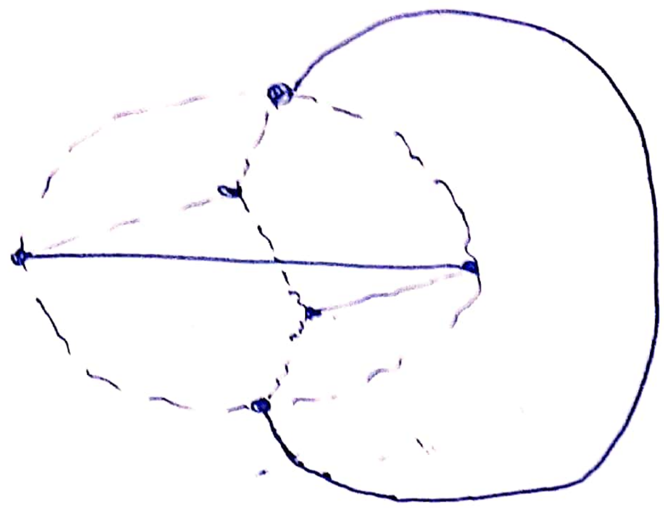
again we have a sub division of $K_{3,3}$ in G [$\Rightarrow\Leftarrow$]

C_3:



Here we have a sub division of K_5 in G [$\Rightarrow\Leftarrow$]

C_4:



Here we have a sub-division of $K_{3,3}$ in G. [$\Longrightarrow \Longleftarrow$]

$\therefore$ All cases lead to a contradiction.

$\therefore$ $\exists$ no such G hence proving Kuratowski's theorem.

Q.E.D.