LOVÁSZ LOCAL LEMMA

Events $A_1, A_2, \ldots, A_n$ $\longrightarrow$ Can these be avoided?

Case 1: A_i's mutually independent and $\mathbb{P}(A_i) < 1 \; \forall i \rightarrow$

$$\mathbb{P}\left(\bigwedge_i \bar{A}_i\right) = \prod_i \left(1 - \mathbb{P}(A_i)\right) > 0 \checkmark$$

Union Bound $\rightarrow$ Case 2: A_i's not necessarily mutually independent but having very small probability $\rightarrow$

$$\mathbb{P}\left(\bigwedge_i \bar{A}_i\right) = 1 - \mathbb{P}\left(\bigvee_i A_i\right) > 1 - \sum_i \mathbb{P}(A_i) > 0 \checkmark$$

But, are cases always so nice?

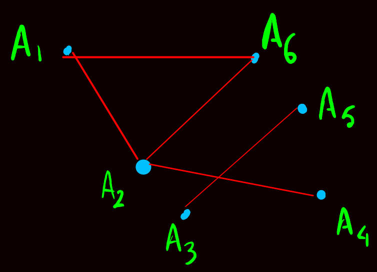
We might be given events with moderately small probabilities, and a certain level of dependence.

So is there a chance of avoiding the events?

- Answered by Lovász - Erdős (1975) & improved by Lovász (1977).

☆ Dependency Graph:

An undirected graph G $\ni V_G = \{A_1, A_2, \ldots A_n\}$ and A_i and A_j are dependent $\Rightarrow \{A_i, A_j\} \in E_G$.



(G)

(define degree for first yrs)

Mind it - the converse is not true and the dependency graph is not unique.

For example, if all A_i's are independent, $E_G = \emptyset$ is valid, whereas a complete K_n is valid in any case but our job is to simplify it as much as possible.

(1) Lemma 1: Lovász - Erdős (1975) Symmetric version:

Conditions: (1) Each event dependent on atmost $d > 0$ other events $(\Delta_G \leq d)$ → Hence Local

Sufficient only (2) $\mathbb{P}[A_i] \leq \frac{1}{4d} \; \forall i \quad \Rightarrow \quad \mathbb{P}[\bigwedge_j \bar{A}_j] > 0$

Here we have a limit d. Notice that when $d = n-1$ (all events are dependent on each other), we get back our union bound.

$\left(n \cdot \frac{1}{4(n-1)} < 1 \iff n \geq 2 \right)$

In order to prove this, we make a claim. You choose an event A_i and take any subset of events T. The probability of A_i happening given that none of the events in T occurred is atmost twice the unconditional probability of A_i occurring.

Mathematically, $\mathbb{P}[A_i \mid \bigwedge_{j \in T} \bar{A}_j] \leq 2\mathbb{P}[A_i] \quad \forall \; T \subseteq V_G$.

We prove it using induction.

Base case: $|T| = 1$.

$$\mathbb{P}[A_1 \mid \bar{A}_2] = \frac{\mathbb{P}[A_1 \cap \bar{A}_2]}{1 - \mathbb{P}[A_2]} \leq \frac{\mathbb{P}[A_1]}{1 - \mathbb{P}[A_2]} \leq 2\mathbb{P}[A_1] \quad \left(\because \mathbb{P}[A_2] \leq \frac{1}{4d} < \frac{1}{2} \right)$$

Inductive case: Let the claim be true $\forall \; |T| \leq k$.

For $|T| = k+1$, we divide $T \to T_1$ (joined to A_1 in G) - nearby events $(\leq d)$

$\searrow T_2$ (others) - far away events

$$\mathbb{P}[A_1 \mid \bigwedge_T \bar{A}_j] = \frac{\mathbb{P}[A_1 \cap \bigwedge_T \bar{A}_j]}{\mathbb{P}[\bigwedge_T \bar{A}_j]} = \frac{\mathbb{P}[A_1 \cap \bigwedge_{T_1} \bar{A}_j \cap \bigwedge_{T_2} \bar{A}_j]}{\mathbb{P}[\bigwedge_{T_1} \bar{A}_j \cap \bigwedge_{T_2} \bar{A}_j]}$$

$$= \left(\frac{\mathbb{P}[A_1 \cap \bigwedge_{T_1} \bar{A}_j \cap \bigwedge_{T_2} \bar{A}_j]}{\mathbb{P}[\bigwedge_{T_2} \bar{A}_j]} \right) \Big/ \left(\frac{\mathbb{P}[\bigwedge_{T_1} \bar{A}_j \cap \bigwedge_{T_2} \bar{A}_j]}{\mathbb{P}[\bigwedge_{T_2} \bar{A}_j]} \right) = \frac{\mathbb{P}[A_1 \cap \bigwedge_{T_1} \bar{A}_j \mid \bigwedge_{T_2} \bar{A}_j]}{\mathbb{P}[\bigwedge_{T_1} \bar{A}_j \mid \bigwedge_{T_2} \bar{A}_j]}$$

$$\leq \frac{\mathbb{P}[A_1]}{1 - \mathbb{P}[\bigvee_{T_1} A_j \mid \bigwedge_{T_2} \bar{A}_j]} \leq \frac{\mathbb{P}[A_1]}{1 - \sum_{T_1} \mathbb{P}[A_j \mid \bigwedge_{T_2} \bar{A}_j]} \leq 2\mathbb{P}[A_1]. \quad \left(\because \underbrace{\sum_{T_1}}_{|T_1| \leq d} \underbrace{\mathbb{P}[A_j \mid \bigwedge_{T_2} \bar{A}_j]}_{\leq 2\mathbb{P}[A_j] \text{ by inductive claim}} \leq 2 \cdot d \cdot \frac{1}{4d} = \frac{1}{2} \right)$$

Hence, claim proved by induction.

Proof of Erdős-Lovász:

$$\mathbb{P}[\bigwedge_{i=1}^{n} \bar{A}_i] = \mathbb{P}[\bar{A}_1]\cdot\mathbb{P}[\bar{A}_2 \mid \bar{A}_1]\cdot\mathbb{P}[\bar{A}_3 \mid \bar{A}_1 \wedge \bar{A}_2]\cdot\ldots\cdot\mathbb{P}[\bar{A}_n \mid \bigwedge_{i=1}^{n-1} \bar{A}_i]$$

$$= (1-\mathbb{P}[A_1])\cdot(1-\mathbb{P}[A_2 \mid \bar{A}_1])\cdot(1-\mathbb{P}[A_3 \mid \bar{A}_1 \cap \bar{A}_2])\cdot\ldots\cdot(1-\mathbb{P}[A_n \mid \bigwedge_{i=1}^{n-1} \bar{A}_i])$$

$$\geq (1-2\mathbb{P}[A_1])\cdot(1-2\mathbb{P}[A_2])\cdot\ldots\cdot(1-2\mathbb{P}[A_n]) > 0 \left(\because \mathbb{P}[A_i] < \frac{1}{2}\ \forall i\right).$$

Lemma 2: Assymmetric Version of Lemma 1:

If $\exists\, x_1, x_2, \ldots x_n \in \mathbb{R}[0,1) \ni \mathbb{P}[A_i] \leq x_i \cdot \prod_{\{A_i,A_j\}\in E_G} (1-x_j)\ \forall i$, then:

$\hookrightarrow$ nearby events T_1

$$\mathbb{P}(\wedge \bar{A}_i) \geq \prod_i (1-x_i).$$

Proof: Similar to proof of lemma 1.

Claim: $\mathbb{P}[A_i \mid \bigwedge_T \bar{A}_j] \leq x_i \quad \forall\, T \subseteq V_G$.

Let $T_1 = \{B_1, B_2, \ldots, B_k\}$.

By Bayes,

$$\mathbb{P}\left(\bigwedge_{T_1} \bar{A}_j \mid \bigwedge_{T_2} \bar{A}_j\right) = \mathbb{P}\left(\bar{B}_1 \mid \bigwedge_{t=2}^{k} \bar{B}_t \wedge \bigwedge_{T_2} \bar{A}_j\right)\cdot\mathbb{P}\left(\bar{B}_2 \mid \bigwedge_{t=3}^{k} \bar{B}_t \wedge \bigwedge_{T_2} \bar{A}_j\right)\cdot\ldots\cdot\mathbb{P}\left(\bar{B}_k \mid \bigwedge_{T_2} \bar{A}_j\right)$$

$$\underset{\text{Inductive assumption}}{\geq} \prod_{T_1}(1-x_j)$$

$$\mathbb{P}[A_1 \mid \bigwedge_T \bar{A}_j] = \frac{\mathbb{P}[A_1 \cap \bigwedge_{T_1} \bar{A}_j \mid \bigwedge_{T_2} \bar{A}_j]}{\mathbb{P}[\bigwedge_{T_1} \bar{A}_j \mid \bigwedge_{T_2} \bar{A}_j]} \leq \frac{\mathbb{P}[A_1]}{\mathbb{P}[\bigwedge_{T_1} \bar{A}_j \mid \bigwedge_{T_2} \bar{A}_j]} \leq \frac{x_i \prod_{T_1}(1-x_j)}{\prod_{T_1}(1-x_j)} = x_i.$$

Hence $\mathbb{P}(\bigwedge_i \bar{A}_j) \geq \prod_i (1-x_i)$ (Expand like we did in lemma 1).

Lemma 3: Lovász Local Lemma (1977)

Conditions: (1) Each event dependent on atmost d other events

Sufficient (2) $\mathbb{P}[A_i] \leq \dfrac{1}{e(d+1)}\ \forall i$. (More relaxed than Lemma 1 for $d>2$).

Proof: Given condition 1 is followed, $x_i \prod_{T_1}(1-x_j)$ attains its maximum when $x_i = \dfrac{1}{d+1}\ \forall i$ (Check using symmetry & calculus).

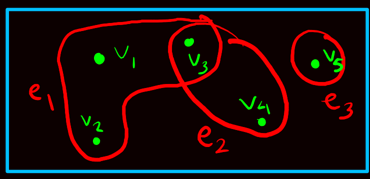
This condition translates to:

$$x_i \prod_{T_1}(1-x_j) = \frac{1}{d+1}\left(\frac{d}{d+1}\right)^{|T_1|} \geq \frac{1}{d+1}\left(\frac{d}{d+1}\right)^{d} \geq \frac{1}{e(d+1)} \geq \mathbb{P}[A_i].\ \left(\because \left(\frac{d}{d+1}\right)^d \downarrow \frac{1}{e} \text{ as } d\to\infty\right).$$

($\because |T_1| \leq d$ & decreasing function) $\Rightarrow$ Conditions for lemma 2 are met.

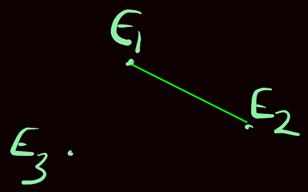
What is a hypergraph?

It's just a generalized graph where an edge is a subset of any number of vertices!



H

See how it might relate to a dependency graph?



Edge set of hypergraph →
Vertex set of dependency graph
Join vertices if edge sets intersect!

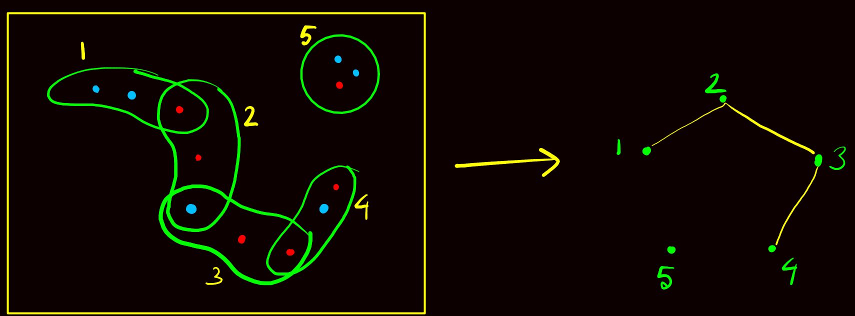
Now, let's go the other way round! (Somewhat)
You have a dependency graph with n vertices $A_1, \dots, A_n$ (events).
Say each event has independent subevents $(A_{11}, A_{12}, \cdots A_{1n_1})$,
$(A_{21}, A_{22}, \cdots, A_{2n_2}), \cdots$ and so on.

We split the vertices into the independent subevents and "encapsulate" them into edge sets. Notice that two events with have a common subevent ($\hookrightarrow \geq 2$ per event) $\Leftrightarrow$ they are dependent $\Leftrightarrow$ the edge sets intersect $\Rightarrow$ the events are joined by an edge in the dependency graph. This gives us our hypergraph!

Applications:

Let in a hypergraph with n edge sets, A_i be the event that the i^{th} edge set is monochromatic (bad event we want to avoid).

R & B



3-uniform hypergraph Dependency graph

$\mathbb{P}(A_i) = ?$

Conditions met?

→ 2-Colouring of k-uniform Hypergraph: Note: $k \geq 2$.

(uniform ↳ Each edge set has k vertices)

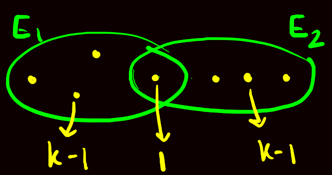
Q1: Given an edge set with k vertices, what's the probability that it's monochromatic?

Choose what to colour the vertices with in 2 ways (R or B).

Then all the vertices will be coloured in that in $\left(\frac{1}{2}\right)^k$ ways.

$\Rightarrow \mathbb{P}(A_i) = 2^{-(k-1)}$.

Q2: Say E_1 & E_2 two edge sets with one intersecting vertex. Given one edge set is monochromatic, what is the probability that the other edge set is also monochromatic?



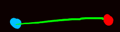
Say E_1 monochromatic $\Rightarrow$ the intersecting vertex fixes the colour of the remaining k-1 vertices of E_2

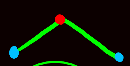
$\Rightarrow 2^{-(k-1)}$.

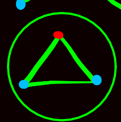
$\therefore P(A_1 | A_2) = P(A_1) \Rightarrow A_1$ & A_2 are independent events.

Q3: What is the minimum number of edges m(k) required for a k-uniform hypergraph to not be two-colourable?

Say $k=2$ (just a graph).

1 edge: 

2 edges: 

3 edges:  not possible.

$\therefore m(2) = 3$

A closed form of m(k) is still open, although $m(k) \geq 2^{k-1}$.

Claim: $m(k) \geq 2^{k-1}$

Take any k-uniform hypergraph H with $m < 2^{k-1}$ edges.

Assign vertex colours randomly.

$\mathbb{P}(e \text{ is monochromatic}) = 2^{-(k-1)}$.

$\mathbb{P}(\text{an edge is monochromatic}) = \mathbb{P}(\bigvee_i A_i) \leq \sum_i \mathbb{P}(A_i) = \frac{m}{2^{k-1}} < 1$.

$\therefore \mathbb{P}(\text{no edge is monochromatic}) = \mathbb{P}(\overline{\bigvee_i A_i}) > 0$.

$\therefore$ H is two-colourable. $\therefore$ All hypergraphs with $m < 2^{k-1}$ edges are two-colourable

What does the Lovász Lemma tell us? (Page 4 - show not necessary)

Thm: Let H be a k-uniform hypergraph $\ni$ each edge intersects atmost d others. Then $d \leq \frac{2^{k-1}}{e} - 1 \Rightarrow H$ is two-colourable.

Pf: Do it yourself (Hint: $\mathbb{P}[A_i] = 2^{-(k-1)}$).

Reference: → Talk on Lovász Local Lemma by Professor Jaikumar Radhakrishnan.

→ Lecture notes of Math 233A by Stanford University

(Will be shared in group).
