EULER'S PENTAGONAL NUMBER THEOREM

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INTRODUCTION

The theory of integer partitions is a subject of enduring interest. A major research area in it's own right, it has found numerous applications and celebrated results. Our today's topic is Euler's Pentagonal Number Theorem.

But before going into that, we need some prerequisites.

SOME PREREQUISITES

1. Ferrer Graphs:

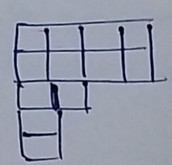
A graphical representation of a partition is useful to explain many theorems and to understand the partition visually.

Ferrer graphs and Ferrer boards are two similar ways of representing an integer partition graphically: the parts of the partition are shown as rows of dots or squares resp.

For example,

$4+4+2+1+1$ is represented by

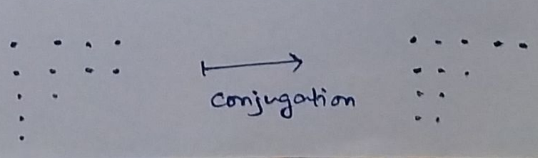
or



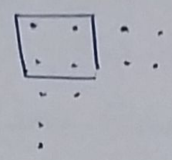
* From a partition, we obtain its conjugate partition by exchanging rows and colums of the Ferrer graph.

It is easy to see that conjugation ~~makes~~ is a bijection between ~~partition of n~~ the set of partitions of n with m parts and set of partition of greatest part m. Hence,

$$p(n \mid m \text{ parts}) = p(n \mid \text{greatest part } m)$$

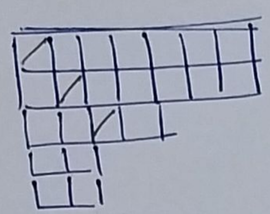


Durfree squares : The largest possible square contained within the left-uppermost corned of the Ferrer board is called the Durfree square of that board. For example, in the partition, $4+4+2+1+1$, the durfree squares is as per picture.



Fronbenius Symbol : From the Ferrer board of a partition, we may construct an entirely new numerical representation of a partition that immediately reveals the size of Durfree square and the conjugate partition. This new representation is Fronbenius Symbol of the partition, and it is constructed as follows,

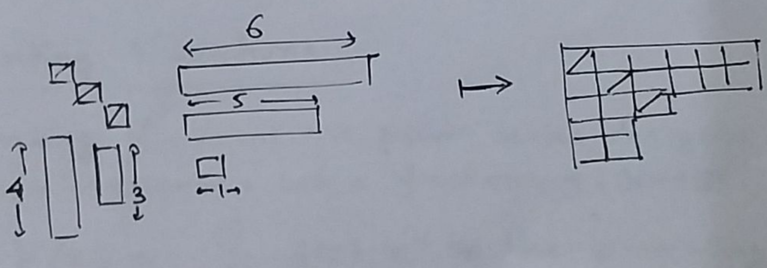
$7+7+4+2+2$



~~The symbol consists of two rows of decreasing size of the Durfree square. The jth entry on the top row con~~

The symbol consists of two rows of decreasing nonnegative integers. The rows are each of length s, where s is the size of the Durfree square. The jth entry on the top rows consists of the number of boxes (or dots) on the jth row of the board to the right of the diagonal. The jth entry on the bottom row consists of the number of boxes in the jth column of the Ferrer board below the diagonal. So the Fronbenius Symbol of the given partition is,

$$\begin{pmatrix} 6 & 5 & 1 \\ 4 & 3 & 0 \end{pmatrix}$$

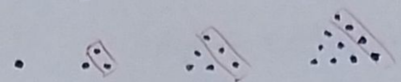


2. n-gonal Numbers :

In mathematics, a polygonal number is a number that counts dots arranged in the shape of a regular polygon. These are 2-dimensional figurate numbers.

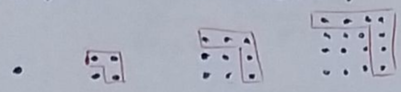
For example,

1. Triangular Numbers:



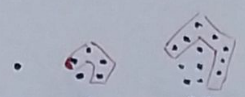
1 3 6 10

2. Square Numbers :



1 4 9 16

3. Pentagonal Numbers:



1 5 12

It is easy to find nth s gonal number, that is

$$P(s,n) = \frac{(s-2)n^2 - (s-4)n}{2} = (s-2)\frac{n(n-1)}{2} + n$$

One can also read about ~~for~~ central polyhedral numbers [These ~~are~~ centered polyhedral numbers are a class of figurate numbers, each formed by a central dot, surrounded by polyhedral layers with constant number of edges. The length of the edge increases by one in each additional layer] Like, tetrahedral, cube, octahedral numbers.

3. Generating Functions :

Generating functions are power series designed to keep track of number sequence. Let, a function $f: \mathbb{N} \to \mathbb{R}$. Then we call

$$F(x) = \sum_{n=0}^{\infty} f(n)\, x^n$$ be the generating function of f.

for example, generating function of binomial numbers $\binom{n}{k}$ is $(1+x)^n$ and generating function of the famous bernoulli numbers is $\frac{x}{e^x - 1}$.

I will give one example to show how much important generating functions are.

Consider, ~~(m+1)~~

$$S_m(n) = \sum_{K=1}^{n-1} K^m$$

We will show that

$$(m+1) S_m(n) = \sum_{K=0}^{m} \binom{m+1}{K} B_K n^{m+1-K}$$

proof: $1 + e^x + e^{2x} + \cdots + e^{(n-1)x} = \dfrac{e^{nx}-1}{e^x - 1} = \dfrac{e^{nx}-1}{x} \cdot \dfrac{x}{e^x-1}$

$$\Rightarrow \sum_{m=0}^{\infty} \frac{x^m S_m(n)}{m!} = \left(n + \frac{n^2 x}{2!} + \frac{n^3 x^2}{3!} + \cdots\right)\left(\sum_{K=0}^{\infty} \frac{B_K x^K}{K!}\right)$$

(Using Tailor series)

$$\Rightarrow \frac{S_m(n)}{m!} = \sum_{K=0}^{m} \frac{B_K n^{(m+1-K)}}{(m+1-K)!\, K!}$$

(comparing coefficients of x^m)

$$\Rightarrow S_m(n) = \frac{1}{m+1} \sum_{K=0}^{m} \frac{(m+1)!}{K!\,(m+1-K)!} B_K n^{(m+1-K)}$$

$$\Rightarrow S_m(n) = \frac{1}{m+1} \sum_{K=0}^{m} \binom{m+1}{K} B_K n^{(m+1-K)}$$

Two variable generating function: Sometimes we need to keep track of more than what number is being partitioned. In many instance we want to have the generating function provide the number of parts of the partition as well.

For example,

$$\prod_{n \in S} (1 + zq^n) = \sum_{n \geq 0} \sum_{m \geq 0} p(n \mid m \text{ distinct parts each in } S) z^m q^n$$

Where S is a ~~set o~~ subset of $\mathbb{N}$.

$$\prod_{n \in S} (1 + zq^n + z^2 q^{2n} + \cdots) = \sum_{n \geq 0} \sum_{m \geq 0} p(n \mid m \text{ parts each in } S) z^m q^n$$

$$\Rightarrow \prod_{n \in S} \frac{1}{1 - zq^n} = \sum_{n \geq 0} \sum_{m \geq 0} p(n \mid m \text{ parts each in } S) z^m q^n$$

[Well, In general let, $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ then generating function of f is $\sum_{n \geq 0} \sum_{m \geq 0} f(n,m) z^m z'^n = F(z, z')$]

4. Laurent Series Expansion :

Let f is a function from ~~com~~ $\mathbb{C}$ to $\mathbb{C}$. The Laurent Series for $f(z)$ around a point $c \in \mathbb{C}$ is given by,

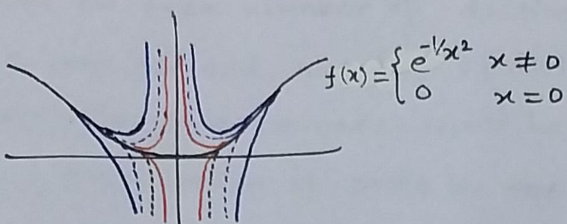
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n$$

This is somehow similar to Taylor series but the difference is, in this case n varies from $-\infty$ to ∞.

One of the use of this is to approximate $f : \mathbb{R} \to \mathbb{R}$ s.t.

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$ at every pt. except 0.

Here the convergent Laurent series expansion of e^{-1/x^2} is $\sum_{n=0}^{N} (-1)^n \frac{x^{-2n}}{n!}$



Laurent approximation of f

EULER'S PENTAGONAL NUMBER THEOREM

Statement of the theorem is,

$p(n \mid \text{even number of distinct parts}) - p(n \mid \text{odd number of distinct parts})$

$= e(n)$ where,

$$e(n) = \begin{cases} (-1)^j & \text{if } n = \frac{j(3j \pm 1)}{2} \text{ for some integer } j \\ 0 & \text{otw.} \end{cases}$$

We will give two proofs of the theorem, one using bijection & other using generating functions.

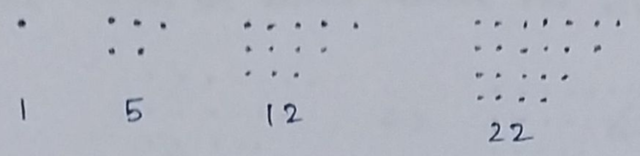
1st Proof:

Remember that, $P(s,n) = \frac{(s-2)n^2 - (s-4)n}{2}$

$$\Rightarrow P(5,n) = \frac{3n^2 - n}{2} = n^2 + \frac{n(n-1)}{2}$$

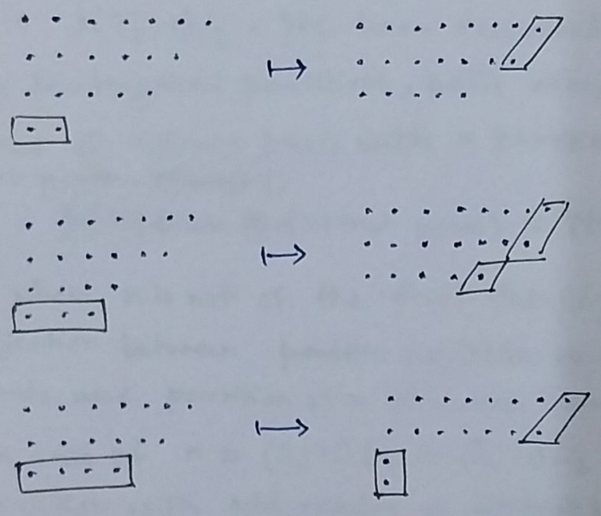
So the jth pentagon consists of the jth triangle standing on top of a rectangle of width j and hight $(j-1)$ (or, $(j-1)$th triangle on jth square)

Now let us turn the pentagons on their side and adjust the dots in the triangle into straight rows so that we obtain Ferrer graphs:



We see that these are Ferrers graphs of certain partition into distinct parts: 1, 3+2, 5+4+3, 7+6+5+4 etc. These particular partition will appear as special cases in the following proof of Euler's pentagonal number theorem. This bijective proof was found of Franklin in 1881.

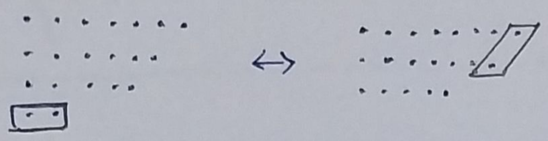
Now we will try to create a bijection between partitions of some integer n into an even number of distinct parts on the one side and partition of n into an odd number of distinct parts on the other side. An invertible transformation would be perfect for this ~~por~~ purpos, that changes the number of parts by one, keeping the distinctness of parts. So as a first idea, what happens if we take the smallest part & distribute its dots on the remaining rows, one on each rows as far as they last? See ~~sketch~~ these examples:



The transformation yields a partition into distinct parts if the rows were atleast as many as the number of dots in the removed parts as in the first two examples. But we must demand something even stronger to make the transformation invertible, since the first two examples resulted in the same partition! Let us find a sensible (means invertibility holds) defination of the inverse ~~cot~~ could ~~do~~ be.

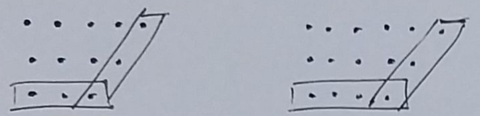
In the inverse direction, we shall take a dot from a few of the largest parts and make a new smallest row. ~~A well known~~

A well defined number of dots to move would be the number of rows that differ by a single dot, starting with the largest row. In other words, we would remove the rightmost diagonal of the graph



When should we remove the shortest row and the rightmost diagonal? The only rule that makes sense is to move the rightmost diagonal if it is shorter that the shortest rows, otw move the latter.

However, there is a case when the above transformation fails to produce a valid Ferrers graph, namely the shortest row actually intersects the rightmost diagonal in the lower right corner of the graph and the rows is the same length or one dot longer than the diagonal.



And these are the only possible case s.t. transformation is invalid. The Ferrers graph in the first case are the pentagons of size $\frac{j(3j-1)}{2}$ dots that we considered at begining and the next case number of dots is $j(3j+1)/2$. We have described that a transformation except these pentagonal partitions, pairs every partition of n into an odd number of distinct parts with a partition of n into an even number of distinct parts. Therefore,

$$p(n \mid \text{even \# distinct parts}) = p(n \mid \text{odd \# distinct parts}) + e(n)$$

[As, when n is not of the form $(3j+1)j/2$ or $(3j-1)j/2$, $\exists$ a ~~perfect~~ bijection between ~~partion~~ partition of n into odd number of distinct parts and partition of n into even number of distinct parts.
In case of $n = (3j+1)j/2$ or $(3j-1)j/2$ where j is odd we get a partition with odd number of distinct parts one extra than even number of distinct parts (the pentagonal partition) so $e(n) = -1$.
Similarly for $n = (3j+1)j/2$ or $(3j-1)j/2$ where j is even we get a partition with even number of distinct parts one ~~etr~~ extra than odd number of distinct parts (the pentagonal partition)] $\square$

~~(Here note that $\nexists$ $m, j \in \mathbb{N}\setminus\{0\}$ s.t. $(3j+1)j/2 = (3m-1)m/2$ (except $j=m=0$) but for $m, j \in \mathbb{Z}$, we~~

(Here note that $\nexists$ $m, j \in \mathbb{N}\setminus\{0\}$ s.t. $(3j+1)j/2 = (3m-1)m/2$ but for $m, j \in \mathbb{Z}$ $j = -m$ satisfies the condition)

2^{nd} Proof:

We want to show that,

$$p(n \mid \text{even \# parts (dist.)}) - p(n \mid \text{odd \# parts (dist.)}) = e(n) \qquad \cdots (1)$$

where $e(n) = \begin{cases} (-1)^j & \text{if } n \text{ is of the form } j\frac{(3j \pm 1)}{2} \text{ for some integer } j \\ 0 & \text{otw.} \end{cases}$

Now we know that,

$$\sum_{n \geq 0} \sum_{m \geq 0} P(n \mid m \text{ dist. part each in } S) z^m q^n = \prod_{n \in S} (1 + zq^n)$$

putting $z = -1$ and $S = \mathbb{N}$ we get

$$\sum_{n \geq 0} \sum_{m \geq 0} P(n \mid m \text{ dist. parts}) (-1)^m q^n = \prod_{n=1}^{\infty} (1 - q^n)$$

$$\Rightarrow \sum_{n \geq 0} \big(P(n \mid \text{even \# dist parts}) - p(n \mid \text{odd \# dist. parts})\big) q^n = \prod_{n=1}^{\infty} (1 - q^n)$$

$\Rightarrow$ Generating function of LHS of (1) is $\prod_{n=1}^{\infty}(1 - q^n)$

And generating function of RHS is

$$\sum_{n=0}^{\infty} (-1)^j q^{\frac{(3j-1)j}{2}} + \sum_{n=1}^{\infty} (-1)^j q^{\frac{(3j+1)j}{2}}$$

$$= \sum_{n=0}^{\infty} (-1)^j q^{\frac{(3j-1)j}{2}} + \sum_{n=-\infty}^{-1} (-1)^j q^{\frac{(3j-1)j}{2}}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^j q^{\frac{(3j-1)j}{2}} \qquad \left(= \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(3j+1)}{2}}\right)$$

So we have to show,

$$\prod_{n=1}^{\infty}(1 - q^n) = \sum_{j=-\infty}^{\infty} q^{\frac{j(3j-1)}{2}} (-1)^j \qquad \left(\text{or } \sum_{j=-\infty}^{\infty} (-1)^j q^{\frac{j(3j+1)}{2}}\right)$$

LEMMA:

JACOBI'S TRIPLE PRODUCT IDENTITY:

$$\sum_{j=-\infty}^{\infty} z^j q^{\frac{j(j+1)}{2}} = \prod_{j=1}^{\infty} (1 - q^j)(1 + zq^j)(1 + z^{-1}q^{j-1})$$

Let, $J(z) = \prod_{n=1}^{\infty} (1+zq^n)(1+z^{-1}q^{n-1})$

Let, Laurant Series Expansion of $J(z)$ around 0 is,

$$J(z) = \sum_{n=-\infty}^{\infty} A_n(q) z^n$$

$$J(zq) = \sum_{n=-\infty}^{\infty} q^n A_n(q) z^n = \prod_{n=1}^{\infty} (1+zq^{n+1})(1+z^{-1}q^{n-2})$$

$$= (1+zq)(1+z^{-1}q^{-1}) J(z)$$

$$= \frac{1}{zq} J(z)$$

$$\Rightarrow J(zq) = \frac{1}{qz} J(z)$$

$$\Rightarrow z J(zq) = \frac{1}{q} J(z)$$

$$\Rightarrow q^n A_n(q) = \frac{1}{q} A_{n+1}(q) \quad \text{(comparing coefficients of } z^{n+1})$$

$$\Rightarrow A_{n+1}(q) = q^{n+1} A_n(q)$$

$$\Rightarrow A_n(q) = q^{\frac{n(n+1)}{2}} A_0(q) \quad \text{(using induction)} \quad \forall n \geq 0$$

$$q^{-n} A_{-n}(q) = \frac{1}{q} A_{-(n-1)}(q)$$

$$\Rightarrow A_{-n}(q) = q^{n-1} A_{-(n-1)}(q)$$

$$\Rightarrow A_{-n}(q) = q^{\frac{n(n-1)}{2}} A_0(q) \quad \text{(using induction)} \quad \forall n \geq 0$$

$$\Rightarrow A_{-n}(q) = q^{\frac{-n(-n+1)}{2}} A_0(q)$$

$$\Rightarrow A_n(q) = q^{\frac{n(n+1)}{2}} A_0(q) \quad \forall n \in \mathbb{Z}$$

Now we have to calculate $A_0(q)$ (the coefficient of z^0 in the expression)

$$A_0(q) = \sum_{\substack{s=0 \\ 1 \leq A_1 < A_2 < \cdots < A_s \\ 0 \leq b_1 < b_2 < \cdots < b_s}}^{\infty} q^{(A_1+A_2+\cdots+A_s)} q^{(b_1+\cdots+b_s)}$$

$$= \sum_{\substack{s=0 \\ 0 \leq a_1 < a_2 < \cdots < a_s \\ 0 \leq b_1 < b_2 < \cdots < b_s}}^{\infty} q^{a_1+\cdots+a_s} q^{b_1+\cdots+b_s} q^s \quad \text{(where } A_i = a_i + 1 \ \forall i \in \{1,2,\ldots,s\})$$

Now we can see that coefficient of q^n is number of ways we can partition n in such way (i.e. as, $n = s + (a_1+\cdots+a_s) + (b_1+\cdots+b_s)$) where $s \geq 1$ and $0 \leq a_1 < \cdots < a_s$ and $0 \leq b_1 < \cdots < b_s$) (Note that $a_s \geq s-1$, $b_s \geq s-1$)

which is equivalent to the Fronbenious symbol $\begin{pmatrix} a_s & a_{s-1} & \cdots & a_1 \\ b_s & b_{s-1} & \cdots & b_1 \end{pmatrix}$

So, coefficient of q^n is $p(n)$.

$$\text{So, } J(z) = \left(\sum_{n=0}^{\infty} p(n) q^n\right)\left(\sum_{j=-\infty}^{\infty} \cancel{(-1)}^j q^{\frac{j(j+1)}{2}}\right)$$

$$= \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right)\left(\sum_{j=-\infty}^{\infty} \cancel{(-1)}^j q^{\frac{j(j+1)}{2}}\right)$$

$$\Rightarrow \prod_{n=1}^{\infty} (1-q^n)(1+zq^n)(1+z^{-1}q^{n-1}) = \sum_{j=-\infty}^{\infty} \cancel{(-1)}^j q^{\frac{j(j+1)}{2}}$$

It is very strong & useful result.

Now using this our proof is just three line proof.

set, z to be $-q^{-1}$ and q to be q^3 so that

$$\prod_{n=1}^{\infty} (1-q^{3n})(1-q^{3n-1})(1-q^{3n-2}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n(n+1)}{2} - n}$$

$$\Rightarrow \prod_{n=1}^{\infty} (1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} \left(= \sum_{n=-\infty}^{\infty} (-1)^n q^{n\frac{(3n-1)}{2}}\right) \qquad \square$$

Source

1. Wikipedia
2. Integer Partitions (Goerge E. Andrews & Kimmo Eriksson)
3. A Primer of Analytic Number Theory (Jeffry Stopple)