Fault-Free Tilings

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Outline

What are Tilings?

In combinatorics, tiling mainly refers to the arrangements of shapes, called tiles, to cover a geometric space without gaps or overlaps. The goal is to find ways in which the space can be partitioned into specific shapes according to certain rules.

The most common example that comes to mind when we think of tilings, is probably the tiling of a $m \times n$ rectangles using dominoes.

Why do we study Tilings?

Though tilings have many applications starting from studying networks to image processing to actual tilings in buildings, we will concern ourselves with the application of tilings in studying the crystalline structure and arrangement of molecules. This example would enable us to understand why fault-free tilings might be interesting to study.

What are Faults?

We assume that we are tiling a $m \times n$ rectangle using some set of tiles.

Definition (Vertical fault)

We say that a given tiling has a *vertical fault* at $x = a$ if for some $1 \leq a \leq n - 1$ the line $x = a$ does not intersect the interior of any tile.

Definition (Horizontal fault)

We say that a given tiling has a *horizontal fault* at $y = b$ if for some $1 \leq b \leq m - 1$ the line $y = b$ does not intersect the interior of any tile.

What are Fault-Free Tilings?

Definition (Vertically fault-free tiling)

A tiling which has no vertical faults is called a *vertically fault-free tiling*.

Definition (Horizontally fault-free tiling)

A tiling which has no horizontal faults is called a *horizontally fault-free tiling*.

Definition (Fault-free tiling)

A tiling is said to be *fault-free* if it is both vertically fault-free and horizontally fault-free.

How are Fault-Free Tilings relevant?

One very interesting example, where fault-free tilings play a crucial role in the study of perfect crystals.

Perfect crystals are the specific atomic or molecular arrangement within a material where every unit cell (the basic repeating structure) is arranged in a fault-free, regular pattern, forming a lattice that extends uniformly throughout the material. This uniformity is key to many of the physical properties that make perfect crystals distinct. Introduction of faults or defects leads to alteration of the physical properties of the crystal.

Talking of fault-free tilings, it is quite astonishing that R.L. Graham was the first person to consider such tilings back in the early 1980s, and since then not much research has been conducted in the field.

Tileability condition

For a $m \times n$ rectangle to be tileable the product mn must be divisible by 2, which implies that at least one of m or n must be even.

The reason why this is a condition for a rectangle to be tileable is fairly obvious. Consider a rectangle $(2k+1) \times (2l+1)$. We consider a grid on this rectangle, where each cell in the grid has unit area. Similarly, we consider that a domino has area of 2 units. Thus, the rectangle has area $2(kl + k + l) + 1$, and so, a unit square remains uncovered by dominoes. Hence, it is not tileable.

A Fault-Free Domino Tiling of 6×5 rectangle

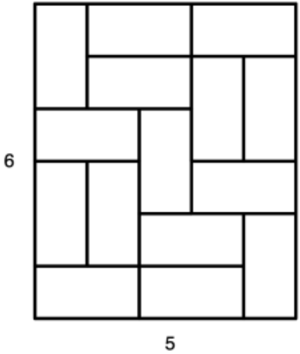

Figure 1: Fault-Free Domino Tiling of 6×5 rectangle

A necessary condition for being Fault-free

The $m \times n$ rectangle must satisfy the condition that $m \geq 5$ and $n \geq 5$ for it to admit a fault-free tiling. We will prove that if $m < 5$ or $n < 5$, then $m \times n$ is not fault-free tileable.

Considering $1 \times q$ and $2 \times q$ rectangles

Figure 2: Fault of $1 \times q$ at $x = 2$

Figure 3: The fault lines in $2 \times q$

Considering $3 \times q$ rectangles I

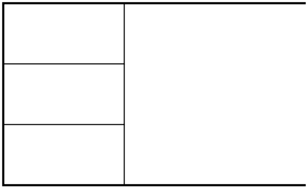

Figure 4: Trying to construct a $3 \times q$ fault-free domino tiling

Considering $3 \times q$ rectangles II

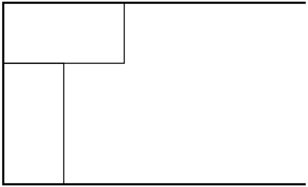

Figure 5: Trying to construct a $3 \times q$ fault-free domino tiling

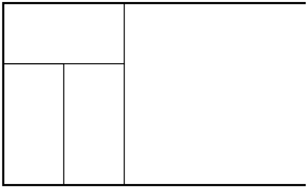

Figure 6: Fault at y=2

Considering $3 \times q$ rectangles III

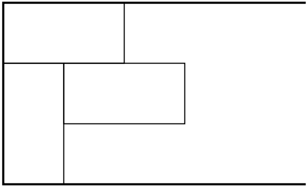

Figure 7: Trying to construct another $3 \times q$ fault-free domino tiling

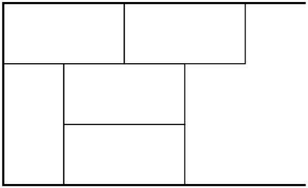

Figure 8: Fault again at y=2

Considering $4 \times q$ rectangles I

Figure 9: Fault at y=2 or x=1

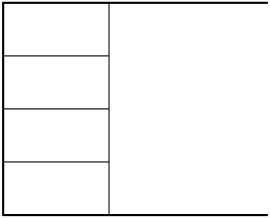

Figure 10: Fault at y=1,2,3 or x=2

Considering $4 \times q$ rectangles II

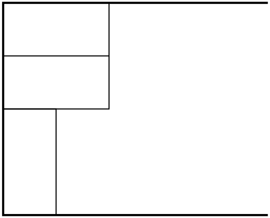

Figure 11: Trying to construct a $4 \times q$ fault-free domino tiling

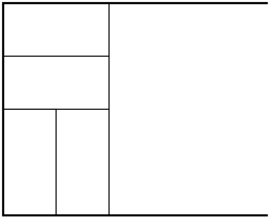

Figure 12: Fault at y=2 and y=3

Considering $4 \times q$ rectangles III

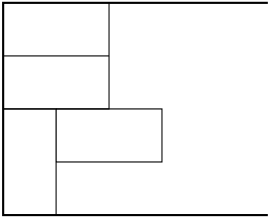

Figure 13: Fault at y=2 and y=3

Figure 14: Again Fault at y=2

Necessary & Sufficient Conditions

Theorem (A)

Let $p, q \in \mathbb{N}$. A fault-free tiling of a $p \times q$ rectangle using dominoes exists if and only if

1. *pq is divisible by 2.*
2. *$p \geq 5$ and $q \geq 5$.*
3. $(p, q) \neq (6, 6)$

As it turns out the very conditions that are necessary for fault-free tilings are also sufficient. (We will talk about the condition $(p, q) \neq (6, 6)$ in a short while.)

Extending the board I

The smallest (p, q) for which a fault-free tiling might exist is $(6, 5)$. We explicitly show one of it's fault-free tilings in Figure 15. We then extend the board along rows and along columns, while always keeping p even without loss of generality, to show that any rectangle $(6 + 2k) \times (5 + 2r + 3l)$ always has a fault-free tiling.

Extending the board II

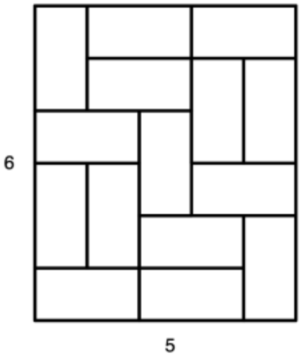

Figure 15: Fault-Free Domino Tiling of 6×5 rectangle

Fissure Breaks

Before extending boards we introduce the concept of fissure breaks.

Definition (Fissure break)

A fissure break is a set of connected line segments going from one side of a board to the opposite side that do not intersect any tiles.

Extending the rows of the board I

Claim (1)

For all $k \in \mathbb{N}$, $(6+2k) \times 5$ has a fault free tiling with a fissure break

$$(0,4) \longrightarrow (3,4) \longrightarrow (3,3) \longrightarrow (5,3)$$

Proof of Claim

We will assume $k \in \mathbb{N}$, and we tile a 6×5 board as seen in Figure 15. We will show that $(6+2k) \times 5$ has a valid fault-free tiling using induction on k.

Base Case: When $k = 1$, we have a 8×5 board. As in Figure 15, we break the tiled 6×5 board along the fissure break, and insert 5 vertically upright tiles in a row as shown in Figure 16. We thus get a fault-free tiling of 8×5 board, which has the fissure break $(0,4) \rightarrow (3,4) \rightarrow (3,3) \rightarrow (5,3)$.

Extending the rows of the board II

Induction Hypothesis: There exists a fault-free tiling of $(6 + 2k) \times 5$ with fissure break $(0, 4) \to (3, 4) \to (3, 3) \to (5, 3)$.

Inductive Step: By the induction hypothesis the same fissure break exists, and so we repeat what we did in the base case. Break along the fissure break and insert 5 vertical tiles. We know that it is a fault-free tiling of $(6 + 2k + 2) \times 5$. Moreover, we have the same fissure break given in our claim statement.

Thus, by induction our claim is true. □

Extending the rows of the board III

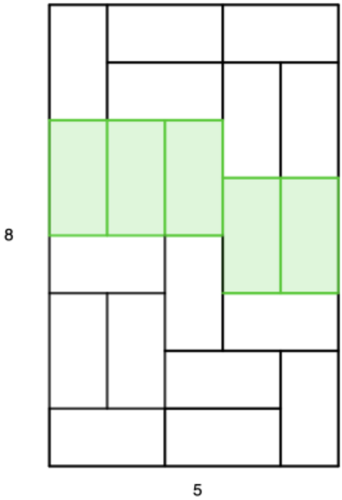

Figure 16: Fault-Free Tiling of 8×5 rectangle using fissure break in 6×5

Extending the columns of the board I

Claim (2)

For all $r, l \in \mathbb{N} \bigcup \{0\}$*, a* $6 \times (5 + 2r + 3l)$ *board has a fault-free tiling with the fissure break*

$$(5 + 2r + 3l - 3, 0) \longrightarrow (5 + 2r + 3l - 3, 4) \longrightarrow (5 + 2r + 3l - 2, 4) \longrightarrow (5 + 2r + 3l - 2, 6)$$

Proof of Claim We will assume that $m, l \in \mathbb{N} \bigcup 0$. We will prove that $6 \times (5 + 2m + 3l)$ is fault-free tileable by a direct proof with cases.

Extending the columns of the board II

Case 1: We have the given fissure break in a tiling of a 6×5 board in Figure 15. We may use this fissure break to add additional tiles to the board. We will add width $2r$ by breaking along the fissure break and insert 6 horizontal tiles as in Figure 17, which is a fault-free tiling. Also, the same fissure break exists, so we can continue to add and break fault lines until we have added width $2r$.

Case 2: We will add the $3l$. We start by adding 3. We break along the given fissure break and add 6 horizontal tiles and 3 vertical tiles as in Figure 18, which is a fault-free tiling. Also, the same fissure break exists, so we can continue to add and break fault lines until we have added width $3l$.

Thus, our claim is true as every $n \in \mathbb{N}$, with $n \geq 2$ can be expressed as $2r + 3l$. □

Extending the columns of the board III

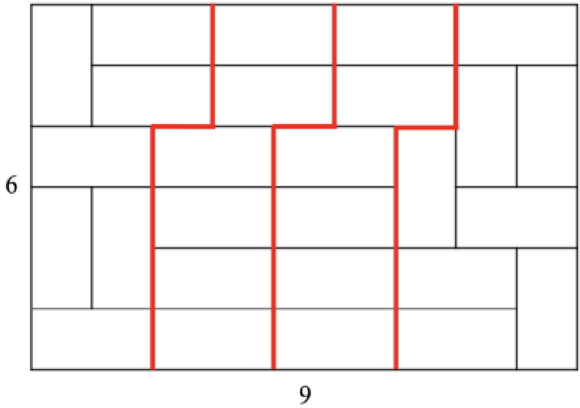

Figure 17: Fault-Free Tiling of 6×9 using fissure break in 6×5

Extending the columns of the board IV

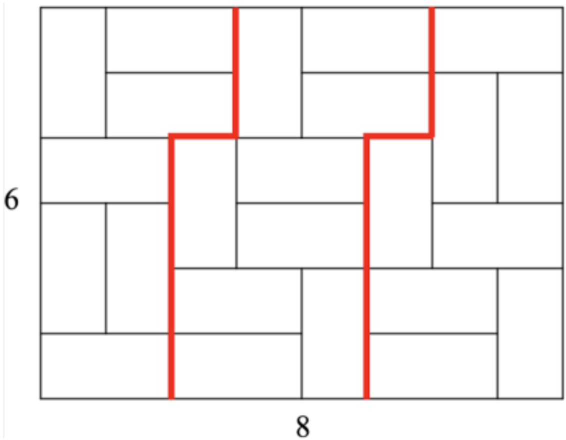

Figure 18: Fault-Free Tiling of 6×8 using fissure break in 6×5

Proof of Sufficiency in Theorem (A) I

Assume p and q are integers such that pq is divisible by 2, $p \geq 5$ and $q \geq 5$, and $(p, q) \neq (6, 6)$. WLOG, assume p is even. We will prove that for any such p and q, a fault-free tiling of a $p \times q$ board exists.

Consider, $p = 6 + 2k$ for some $k \in \mathbb{N} \bigcup 0$ and $q = 5 + 2r + 3l$ for some $r, l \in \mathbb{N} \bigcup 0$. This can be done for any even integer $p \geq 6$ and any integer $q \geq 5$ for $q \neq 6$. We start with the tiling of 6×5 rectangle as given in Figure 1.

Proof of Sufficiency in Theorem (A) II

By Claim (1), we know that we can add any even number of height by keeping the tiling fault-free. So we add height $2k$. This gives a fault-free tiling of a $(6 + 2k) \times 5$ board. This tiling will have the fissure break described in Claim (2). So, we insert tiles to increase the width by $2m + 3l$, and obtain a fault-free tiling as evident from Claim (2).

In the case where $q = 6$ we know $p \neq 6$. Using the technique above we can construct a fault-free tiling of a $6 \times p$ board and rotate it to have a tiling of $p \times 6$ board. Thus, the conditions given are sufficient to have a fault-free tiling. □

Generalization

Conditions for fault-free tileability of rectangles using dominoes naturally gives rise to a question:

What are the conditions for fault-free tileability of rectangles using tiles other than dominoes?

In 1981 *R.L.Graham*, gave the conditions for which a $p \times q$ rectangle admits a fault-free tiling using $a \times b$ tiles. The paper by Graham, was the very first paper that formally dealt with the problem of fault-free tilings.

Analogue to the Tileability condition

For a $p \times q$ rectangle to be tileable using $a \times b$ tiles, each of a and b must divide at least one of p or q.

An obvious question that might arise is that *"Why is the tileability condition not ab divides pq?"*. To check this, we can take $(a, b) = (14, 15)$ and $(p, q) = (20, 21)$. We can observe that the rectangle $p \times q$ is not tileable using $a \times b$, although it satisfies the condition ab divides pq.

Analogue to Fault-Free Tileability condition

There should be an analogue to the earlier second condition that required $p, q \geq 5$. However, the corresponding condition is unexpected. It states that:

Each of p and q must be able to be expressed as a sum of $xa + yb$ with positive integers x and y in at least two ways.

Placing a stack of horizontal tiles or a stack of vertical tiles guarantees the existence of a fault. Thus, this condition gives us the freedom to place the tiles, such that, we do not always have to place the same number of tiles horizontally and vertically.

Necessary & Sufficient Conditions

Theorem (B)

A fault-free tiling of a $p \times q$ rectangle with $a \times b$ exists (where we assume $pq > ab$ and $\gcd(a, b) = 1$ if and only if

1. *Each of a and b divides at least one of p or q.*
2. *Each of p and q can be expressed as $xa + yb$, $x, y > 0$, in at least two ways.*
3. *For $(a, b) = (2, 1)$, $(p, q) \neq (6, 6)$.*

Proof Sketch of Sufficiency I

Consider $p = x_1 a + y_1 b = x_2 a + y_2 b$ and $q = x_3 a + y_3 b = x_4 a + y_4 b$. Without loss of generality we assume that $a > b$, and also $x_1 > x_2$, $y_1 < y_2$, $x_4 > x_3$, $y_3 > y_4$.

Now we refer to Figure 19, where we can place horizontal tiles in the rectangle $y_1 b \times x_3 a$. Similarly we tile the rectangle $x_1 a \times y_4 b$ using only vertical tiles. In a similar manner we also tile the other rectangles $y_2 b \times x_4 a$ and $x_2 a \times y_3 b$. Owing to the fact that $gcd(a, b) = 1$, we get that the fault lines are removed as the probable fault line tries to move through one of these subrectangles to the next, both vertically and horizontally.

Proof Sketch of Sufficiency II

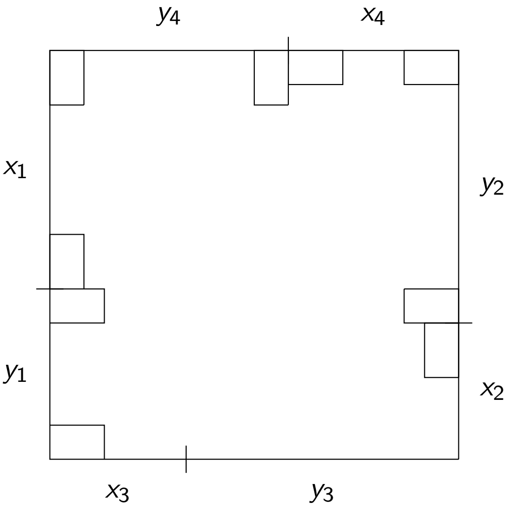

Figure 19: A visualisation of Proof sketch argument

Further Progress

Though the first encounter with fault-free tilings dates back to the 1980s, we still do not know much about them and they turn out to be hard to study. The only progress that has been done in fault-free tilings is:

1. Aanjaneya, M., & Pal, S. P. (2006). Faultfree Tromino Tilings of Rectangles. arXiv preprint math/0610925.
2. Montelius, E. (2019). Fault-Free Tileability of Rectangles, Cylinders, Tori, and Möbius Strips with Dominoes. arXiv preprint arXiv:1912.04445.
3. Alabi, O. J., & Dresden, G. (2021). Fault-Free Tilings of the $3\times$ n Rectangle With Squares and Dominos. Journal of Integer Sequences, 24(2), 1-11.

The Obvious Next Questions I

1. One obvious step is finding generating functions for number of fault-free tilings of some particular $p \times q$ rectangle using $a \times b$ tiles.
2. Another possibility is to find conditions for fault-free tilings of rectangles (and other shapes) using a set of tiles rather than just one or by using tiles that are not of the form $a \times b$.

The Obvious Next Questions II

3. Another question that naturally arises is why are there no fault-free tilings of the 6×6 rectangle using dominoes, and why is it the only anomaly.
4. Another question that arises from an observation is that the number of fault-free tilings seems to drop at even q for $p = 6$ up to $q \leq 10$. It goes as follows: $(0, 0, 0, 0, 6, 0, 124, 62, 1646, 1630, 18120, 25654, 180288, 317338, ...)$. Similarly the number of fault-free tilings takes a dip at $q = 6$ for $p = 8$. However, in all other cases the number of fault-free tilings steadily increases (if a tiling exists in that case) as q increases while p remains fixed. No reason is known for this behaviour.

References

1. R. L. Graham, Fault-free tilings of rectangles, in D. A. Klarner, ed., The Mathematical Gardner: A Collection in Honor of Martin Gardner, Wadsworth, 1981, pp. 120–126.
2. Dettling, T. E., & Goetzinger, A. (2022). Fault-Free Tilings.