A Tale of Two Cities: With Coins and Tiles

Hrishik Koley

Indian Statistical Institute, Bangalore

LIMIT Camp 2025

Contents

What is Combinatorics?

Combinatorics is the branch of mathematics that deals with counting, arrangements, patterns, and structures. It's where questions are easy to ask but sometimes very hard to answer.

Examples

- Assume five people have five books. In how many ways can you redistribute them such that no one gets the book that they had originally?
- How many ways can you use rhombus-shaped tiles to tile a hexagon such that no two tiles overlap?
- Can you colour the vertices of a graph using k many colours such that no two adjacent vertices have the same colour?
- What is the minimum number of integers that a set must have such that there exists a subset of n many elements that sum up to a multiple of n?

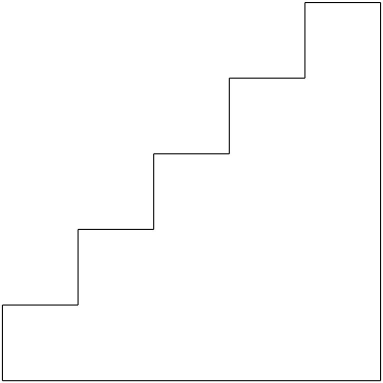
A Warm-Up Riddle

Question

In how many ways can you climb a staircase with 5 steps if you can take either 1 step or 2 steps at a time?

Answer: Well, there are 8 ways to do it. And a fun fact for you guys, that is exactly the 6^{th} Fibonacci number.

Visual Aid



1. $1+1+1+1+1$
2. $1+1+1+2$
3. $1+1+2+1$
4. $1+2+1+1$
5. $2+1+1+1$
6. $2+2+1$
7. $2+1+2$
8. $1+2+2$

Let's Roll

Transition

"Just like this simple riddle, today's talk explores how simple games — like firing coins from vertices or tiling floors — can connect to deep and beautiful mathematics."

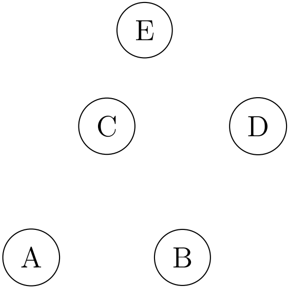
The Kingdom of Graphonia

Story Time

- In the land of Graphonia, cities are connected by roads.
- The king of Graphonia has a strange rule called **Coin Exchange**: *"Whenever a city has enough gold coins to send one to each neighbor, it must do so!"*
- The result is a chain of coin movements. Sometimes which could potentially lead to utter chaos.
- What happens next? That's what we'll discover.

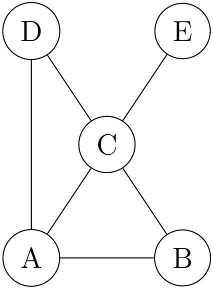
Setting up Graphonia on a Graph I

- **Cities** = Vertices on a graph.



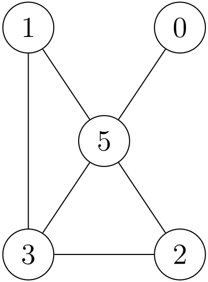
Setting up Graphonia on a Graph II

- **Roads** = Edges of the graph.



Setting up Graphonia on a Graph III

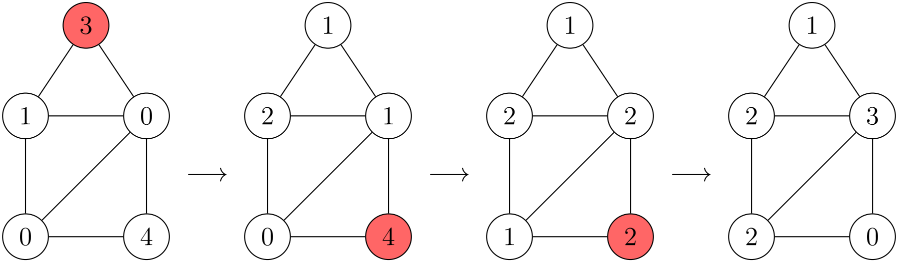
- **Coins** = Chips placed on vertices.



Setting up Graphonia on a Graph IV

- **Number of roads from each city** = Degree of corresponsing vertex
- **Firing Rule**: A city fires if it has at least as many coins as neighbors, sending 1 coin to each

Getting our hands dirty

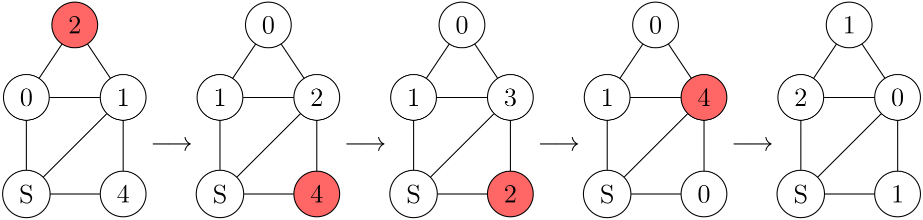


Introducing the Sink

Progressing the Story

- Imagine now that the there is a path to the main city from some of the other cities.
- The main city is where the king lives. The king has another rule called **Tax**.
- **Tax** rule says that: *"All cities other than the main city must follow the coin exchange rule, that is, no matter how many coins the main city has it never has to send coins to it's neighbours."*
- We refer to this main city as the **sink vertex**. We denote it with **S**.

Getting our hands dirty again



JS Sandpile Simulation

- Now time for some simulation.
- For our simulation, we take a square cluster of cities, essentially a grid.
- Each small square cell is a city.
- If two square cells share a side they have an edge connecting them.
- All square cells sharing a side with the boundary of the grid are sink vertices.

Simple Ideas I

Goal

Our goal is stabilization, always was always will be. We want to reach a state where no vertex can fire, be it with the sink vertex or without.

Simple Ideas II

The Obvious

- Assume the graph has no sink vertex.
- We continue the firing process till we attain stabilization.
- The total number of chips in the graph stays constant throughout the firing process.

Simple Ideas III

Simple yet Elegant

- At any point during the firing process we may have more than one vertex that have more coins than it's degree.
- No matter what order we follow to fire these vertices we always get the same result, that is, the same configuration.
- We call this the abelian property.

Abelian Property Proof Sketch

Proof Idea

We use classic induction trick assuming that the firing sequence eventually ends (or stabilizes).
Base Case: We have two vertices that we can fire. The result after firing both is independent of order because coin movement is local and conservative.
Now you can formulate your induction hypothesis for $n-1$ vertices and extend it to n vertices. And the classic induction conclusion proves the abelian property

Food for Thought

Question

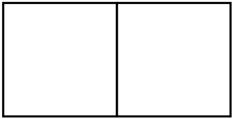
Construct a graph and assign coins to each vertex for which the firing process never stops.

The Temple of Tessellatia

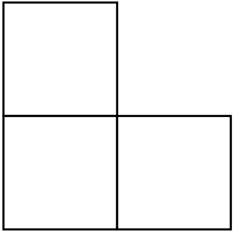
New City, New Story

- In Tessellatia, the floor of a temple has a 4×4 block in the middle tiled with dominoes, that is, 2×1 rectangle tiles.
- A magician visits the temple. He is allergic to such domino tiles and gets furious.
- In a fit of rage, he magically removes a 1×1 square from the corner.
- He also turned all those domino tiles into L-shaped ones, leaving the symmetric harmony of the center stage in tatters.
- Our goal: tile this new shape using these L-shaped tiles.

Introducing the Tiles (New and Old)

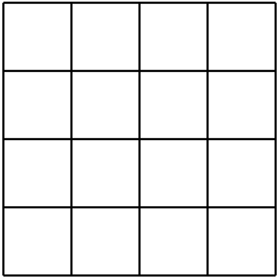


Domino tile

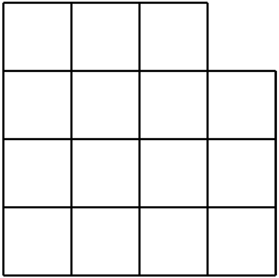


L-shaped tile

Introducing the Stage (New and Old)

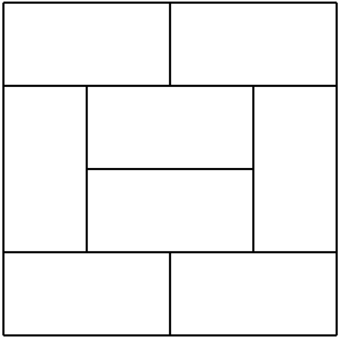


Full 4×4 Grid

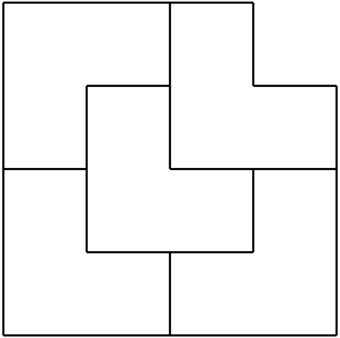


4×4 Grid with
Corner Square Missing

The Tiling of the Stage (New and Old)

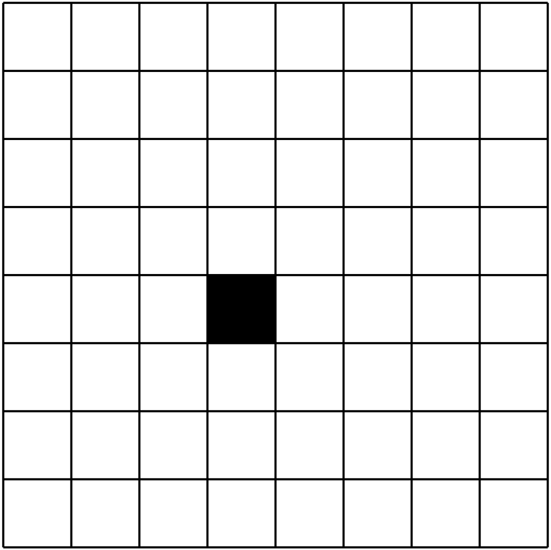


Full 4×4 Grid with Dominoes

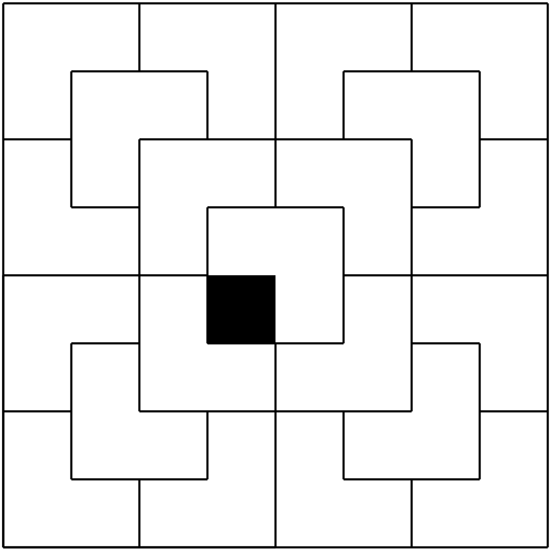


4×4 Grid with Corner Square Missing with L-shaped Tiles

Removing a Tile from the Centre



Tiling the New Shape



Problem at Hand

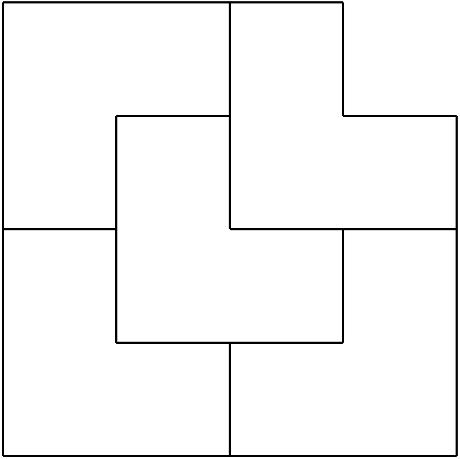
The Bigger, The Better

Now we have a grid of size $2^n \times 2^n$ with one 1×1 square missing. Can we tile this using L-shaped tiles? The answer is yes. But the question is "How?"

Recursive Tiling (The Baby Steps)

- **Base Case:** 2×2 grid with 1 tile missing. Well, it's simple use 1 L-tile.
- For 4×4:
- Divide into four 2×2 squares.
- Place L-tile in the center covering the 3 squares of quadrants that don't contain the missing tile.
- Recurse on each smaller quadrant.

Visualizing Again



General Recursive Strategy

- For grid of size $2^n \times 2^n$:
- Divide into four $2^{n-1} \times 2^{n-1}$ quadrants.
- One quadrant has a missing tile.
- Place L-tile in center to balance remaining three quadrants.
- Recurse.

Proof Sketch of Our Recursive Idea

Proof Idea

We induct on n.
For $n = 1$, we already know the solution for 2×2 grid.
Formulate the induction hypothesis for $2^n \times 2^n$
Now place a L-tile in the center of $2^{n+1} \times 2^{n+1}$ covering squares of those $2^n \times 2^n$ quadrants that do not have a missing square.
Each sub-grid now has one missing tile. And BOOM!! we are back to our $2^n \times 2^n$ case though four of them. So, we repeat the same.

More Food for Thought

Question

What happens if more than one square is missing? Can we still tile it using L-shaped tiles?

Conclusion I

The Vast World of Combinatorics

Combinatorics is magical because: It uses very simple rules to uncover surprisingly deep truths. Sometimes, a problem that looks extremely complicated can be solved using just clever counting. Other times, a problem that looks easy might secretly require algebra, probability, or recursion to crack.

It's a junction to many parts of math — algebra, geometry, probability, number theory — all woven through clever puzzles and logic.

Conclusion II

Final Thought

"What looks like a game can become deep mathematics. Combinatorics shows us that patterns, puzzles, and proofs can all begin with play."