Tribone Tilings: Honeycombs to Benzels

Hrishik Koley

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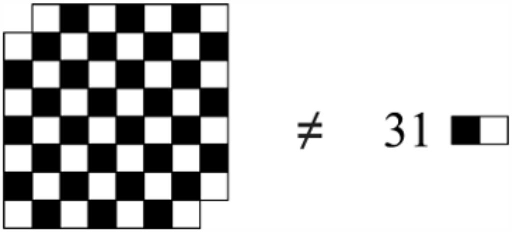
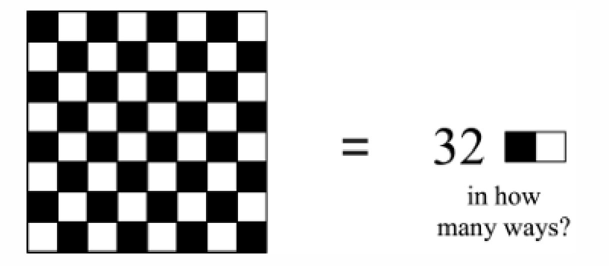
A classic puzzle



If two opposite 1-by-1 corner-squares of an 8-by-8 square are removed, the remaining region cannot be tiled by 1-by-2 and 2-by-1 rectangles.
Each tile has one white square and one black square. But the region being tiled has unequal numbers of black and white squares.

A less famous problem



However, if we prevent ourselves from removing the corner squares then we have a fairly simple tiling problem in our hands.

Origin of Domino/Dimer Tilings

The dimer problem was first considered by physicists, at the intersection of graph theory and enumerative combinatorics.

The classical domino tiling problem considers a rectangle of dimension $m \times n$, divided into a grid. The problem at hand is to tile the entire rectangle with vertical and horizontal dominoes (i.e. 1×2 and 2×1 rectangles, respectively), without any overlap between any two dominoes.

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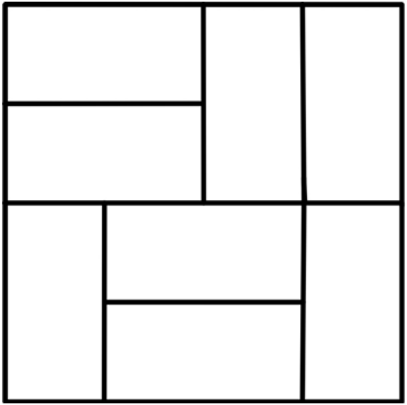
A Common Example

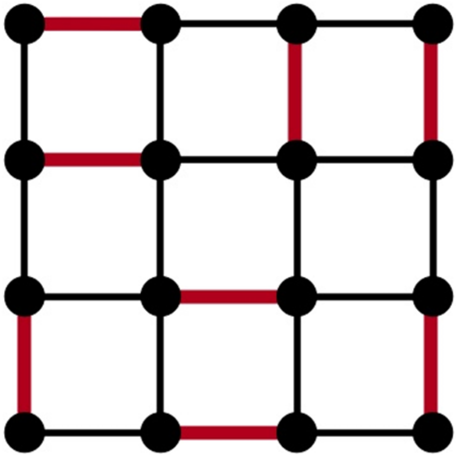


Consider this 4×4 rectangle. It is fairly simple to tile this structure using 1×2 and 2×1 tiles. But what if we have a $m \times n$ rectangle with fairly large enough m and n. It becomes difficult to check and try to find a tiling. Thus, we consider the analogue of this problem with a perfect matching or dimer cover problem in a graph.

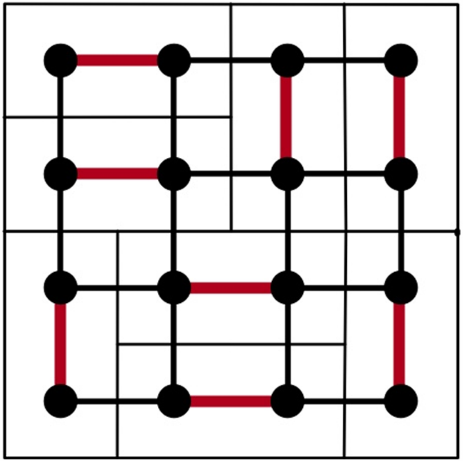
Dimers and Dimer Covers

A dimer in a graph $G = (V, E)$ is just an edge $e \in E$.
A dimer cover of a graph (V, E) (aka a perfect matching) is a set $E' \subseteq E$ of edges with the property that each $v \in V$ belongs to exactly one $e \in E'$.

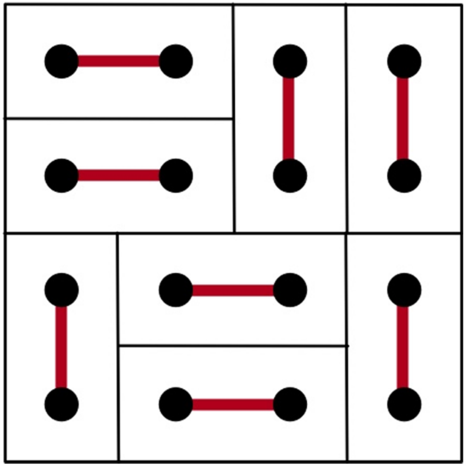
From Dimers to Dominoes I



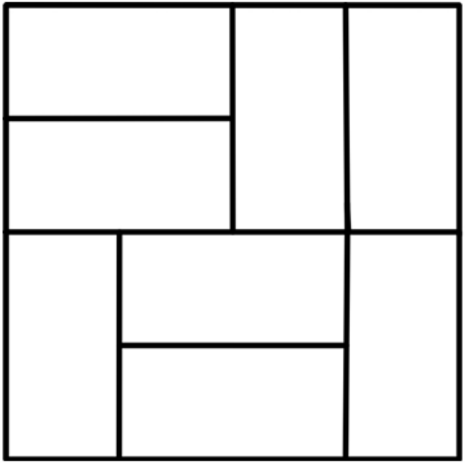
From Dimers to Dominoes II



From Dimers to Dominoes III



From Dimers to Dominoes IV



Tiling Problems that concern us

From a enumerative combinatorial point of view, we have two problems that mainly concern us. Given any shape, we consider two questions at large:

- Is the shape tileable using a given set of tiles?
- How many ways are there to tile the shape given a set of tiles?

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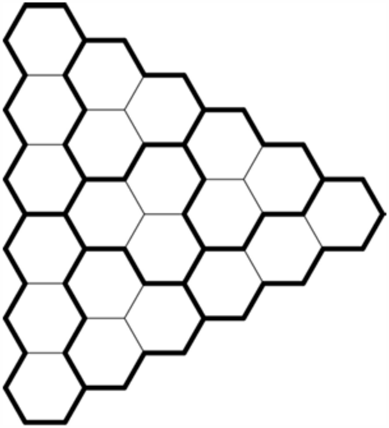
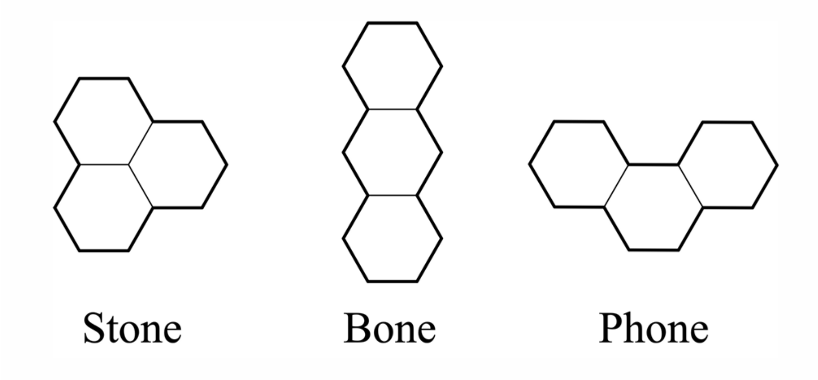
What are Honeycomb Tilings?

Honeycombs or roughly triangular regions in hexagonal lattice, represented by T_n are structures with 3 boundaries of the roughly triangular region, each formed by joining n hexagons, and the entire being occupied by a T_{n-3} honeycomb, where T_1 is a single hexagon.



Introducing the Tiles

Considering any tiling problem of a shape is futile, if we are not aware of the set of tiles that we are allowed to use. Benzel tilings and honeycomb tilings allow us to use trihex tiles. Trihex tiles are basically three hexagons joined edge-to-edge. Owing to symmetry, it is fairly obvious that we are thus allowed to use three types of tiles. For the naming of these tiles, we stick with Propp's and Thurston's convention.



The Conjecture

Theorem (Conway, Lagarias)

It is impossible to tile the honeycomb T_n using only tribone tiles.

Some Definitions and Ideas I

The square lattice in $\mathbb{R}^2$ consists of lattice points, edges, and cells. A *lattice point* is a member of $\mathbb{Z}^2$. Two lattice points are *neighbors* if they are at distance one from each other, so each lattice point has exactly four neighbors.

An *edge* is a line segment connecting two neighboring lattice points; it is either horizontal or vertical.

A *cell* is the set of all points making up the interior and boundary of a square of area one having its four vertices at lattice points.

Some Definitions and Ideas II

A *directed path* P in the square lattice consists of a sequence of directed edges specified by a sequence of lattice points $\{(x_i, y_i) : 0 \leq i \leq n\}$, where the i^{th} directed edge connects (x_{i-1}, y_{i-1}) to (x_i, y_i). It is *closed* if $(x_0, y_0) = (x_n, y_n)$.

A directed path is *simple* if no edge appears twice and if it does not cross itself, where we say a path *crosses itself* if there is $0 < i < n$ and $j \neq i$, such that $(x_i, y_i) = (x_j, y_j)$ and the two edges from (x_{i-1}, y_{i-1}) to (x_{i+1}, y_{i+1}) consist of either two horizontal or two vertical edges.

Some Definitions and Ideas III

A *region* R is a finite connected set of closed cells.
The *topological boundary* $\partial(C)$ of a cell C consists of its four edges, oriented counterclockwise. The boundary $\partial(R)$ is formed by taking the set of all edges in $\partial(C)$ for all cells C in R, and discarding any edges that occur twice with opposite orientations.

A region R is *simply connected* if its complement $\overline{R} = \mathbb{R}^2 - R$ is connected and if its boundary edges can be ordered to form a simple closed path. We call a region to be *connected* if it cannot be expressed as the disjoint union of two regions.

Some Definitions and Ideas IV

A simple closed path bounding a simply connected region R is uniquely specified by its first edge e; we call such a path an *oriented boundary of* R *with leading edge* e and denote it by $\partial R(e)$. The first vertex in $\partial R(e)$ is called the *base point* of $\partial R(e)$.

Some Definitions and Ideas V

A *tile type* consists of the set of all translations of a tile.

A *tiling problem* consists of a region R and a set Σ of tile types. A region R can be tiled by Σ if there exists a set of tiles in Σ that cover each cell of R exactly once.

Some Definitions and Ideas VI

We describe directed paths in the square lattice by words in the free group $F = < A, U >$ on two generators, where $A = across$, and $U = up$. We read each word $W(\partial R(e))$ from right to left to obtain the counterclockwise directed path $\partial R(e)$ from a given base point e.

Given an oriented boundary $\partial R(e)$ of a simply connected region R we let $\partial R(e)$ also stand for the word $W(\partial R(e))$ in F. The words $\{\partial R(e) : e$ a counterclockwise oriented edge of $\partial R\}$ are cyclic permutations of each other, and hence are all conjugate in F.

Thus, the *combinatorial boundary* $[\partial R]$ of a simply connected region R is the conjugacy class in F containing all oriented boundaries $\partial R(e)$ of R, that is -

$$[\partial R] = \{W \partial R(e) W^{-1} : W \in F\}$$

Cycle group, Tile group, and Tile Homotopy group

The *cycle group* C is the subgroup of the free group F consisting of all words associated to closed directed paths in the square lattice.

The *tile group* $T(\Sigma)$ is a subgroup of F that contains all the boundaries of the tiles in the set of tiles $\Sigma = \{R_i\}$.

$$T(\Sigma) =< W\partial R_i(e_i)W^{-1} : W \in F, 1 \leq i \leq m >,$$

where $\partial R_i(e_i)$ is an oriented boundary of R_i.

The tile group $T(\Sigma)$ is contained in the cycle group C. We call the quotient group $h(\Sigma) = C/T(\Sigma)$ the *tile homotopy group*.

Tileability I

Theorem

A simply connected region R has a tiling by tiles in a set Σ only if the combinatorial boundary $[\partial R]$ of R be contained in the tile group $T(\Sigma)$.

The idea is intuitive enough and so the proof is not included in the talk. However one can look at the proof from Conway and Lagarias' paper.

Tileability II

Theorem

Any honeycomb T_n is tileable using triangle tiles and tribone tiles only if $n \equiv 0$ or $2 \pmod 3$.

Proof.

The total number of hexagonal cells in T_n is always $N = \frac{n(n+1)}{2}$. Now, as each tile is a combination of 3 hexagons, so N must be divisible by 3. Thus, for $3 \mid N$, $n \equiv 0$ or $2 \pmod 3$. □

Hexagonal Lattice to Square Lattice

The problem at hand can now be easily converted to a tiling problem on the square lattice. For the honeycomb T_n on the hexagonal lattice to a staircase T'_n on the square lattice.

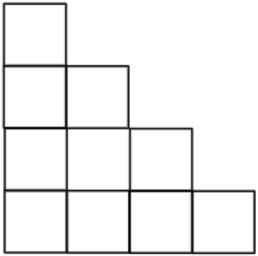


Figure 1: Staircase T'_4

The combinatorial boundary of any staircase T'_n is thus given by the representative word $\partial(T'_n) = A^n U^{-n}(A^{-1}U)^n$.

Conversion of Tribone Tiles

As we shift from the hexagonal lattice to the square lattice the tiles that we use must also change. We notice that the single tribone tile in the hexagonal lattice decompose into 3 different tiles in the square lattice which we will call R_1, R_2, and R_3, respectively.

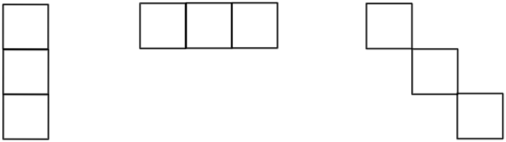


Figure 2: The tiles R_1, R_2, and R_3, respectively

The combinatorial boundary of these tiles are thus, $\partial(R_1) = AU^{-3}A^{-1}U^3$, $\partial(R_2) = A^3U^{-1}A^{-3}U$, and $\partial(R_3) = (AU^{-1})^3(A^{-1}U)^3$.

Cayley Diagram I

The Cayley diagram $\mathcal{G}(F_g/K)$, is a labelled graph with directed edges associated to the quotient group $G = F_g/K$ of the free group F_g on g generators, where K is a normal subgroup of relations. The Cayley diagram of G has a vertex corresponding to every element $W \in G$, and for each generator S_i of F_g there is a directed labelled edge i from W to S_iW. In particular every vertex in a Cayley diagram has $2g$ edges incident on it, with g many directed inwards and g many directed outwards.

Cayley Diagram II

To understand K simply, we consider $\overline{\mathcal{G}}(F_g/K)$, which is the undirected labelled graph corresponding to $\mathcal{G}(F_g/K)$. Associate words W from F_g to directed paths on the edges of undirected graph $\overline{\mathcal{G}}(F_g/K)$. We say, that a word W is in K iff it corresponds to a closed path in $\overline{\mathcal{G}}(F_g/K)$ starting from the identity vertex I.

A special subgroup $\mathbf{H}$

The special subgroup $\mathbf{H}$ of F_2 is defined by the property that it has the associated quotient group $G = F_2/\mathbf{H}$, whose undirected Cayley diagram $\overline{\mathcal{G}}(F_2/\mathbf{H})$ is an infinite planar graph, that tiles the plane using triangles and hexagons.

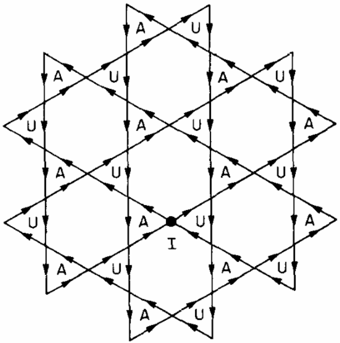


Figure 3: Cayley diagram $\mathcal{G}(F_2/\mathbf{H})$

Some facts related to **H**

The subgroup of relations **H** is given by

$$\mathbf{H} = N(< A^3, U^3, (U^{-1}A)^3 >)$$

The relevance of **H** is however established through the following theorem.

Theorem

*The tile group $T(\Sigma)$, and the combinatorial boundaries $[\partial T_n]$ for $n \equiv 0$ or 2 (mod 3) are all contained in **H**.*

A homomorphism from $\mathbf{H}$ to $\mathbb{Z}$ I

The winding number $w(P; s)$ of a closed directed path P in $\overline{\mathcal{G}}$ around s counts the number of times P encloses the cell s in the counterclockwise direction.
The quantity $w(P; s)$ is additive in nature, that is, $w(P_2 P_1; s) = w(P_2; s) + w(P_1; s)$.
Now, for any finite or infinite set of cells S, we have $w(P; S) = \sum_{s \in S} w(P; s)$.

A homomorphism from $\mathbf{H}$ to $\mathbb{Z}$ II

Using the notion of $w(P; s)$ we define a homomorphism $\phi : \mathbf{H} \longrightarrow \mathbb{Z}$, with $\phi(V) = w(V; S)$, where S is the set of all hexagons in the Cayley diagram $\overline{\mathcal{G}}(F_2/\mathbf{H})$.

Answering the conjecture I

Using definition of ϕ, we can easily observe that $\phi(\partial(R_1)) = 0$, $\phi(\partial(R_2)) = 0$, and $\phi(\partial(R_3)) = 0$. We will check that $\phi(\partial(R_3)) = 0$, and the rest is done is a similar method.

Now as we know that $W\partial(R_i)W^{-1}$ is a conjugate of $\partial(R_i)$, so $\phi(W\partial(R_i)W^{-1}) = \phi(\partial(R_i))$.

Answering the conjecture II

Similarly, a computation using $\partial(T_n') = A^n U^{-n} (A^{-1} U)^n$ in the Cayley diagram gives us that $\phi(\partial(T_n')) = \left[\frac{n+1}{3}\right]$.

Now we asssume that $\partial(T_n')$ belongs to the tile group $T(\Sigma)$, then

$$\partial(T_n') = \prod_{i=1}^{m} W_i \partial(R_{k_i})^{\epsilon_i} W_i^{-1},$$

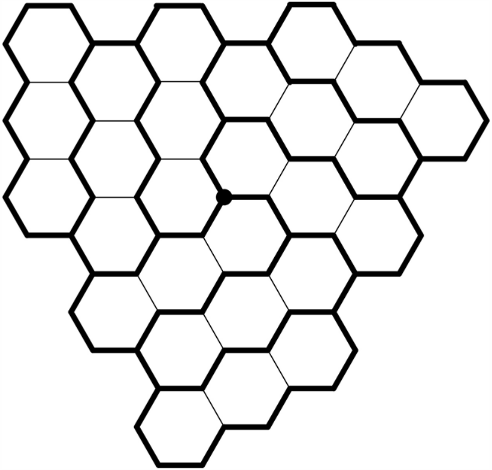
where $W_i \in F_2$, $k_i \in \{1, 2, 3\}$, and $\epsilon_i = 0$ or 1.

Thus, $\phi(\partial(T_n')) = \sum_{i=1}^{m} \epsilon_i \phi(\partial(R_{k_i})) = 0$.

This is a contradiction and hence, $\partial(T_n')$ is not in $T(\Sigma)$, and thus, T_n is not tileable using tribones. □

What are Benzel Tilings?

Benzels form a two-parameter family, with parameters a and b satisfying $a \leq 2b - 2$ and $b \leq 2a - 2$. Here is the $5, 7$-benzel:



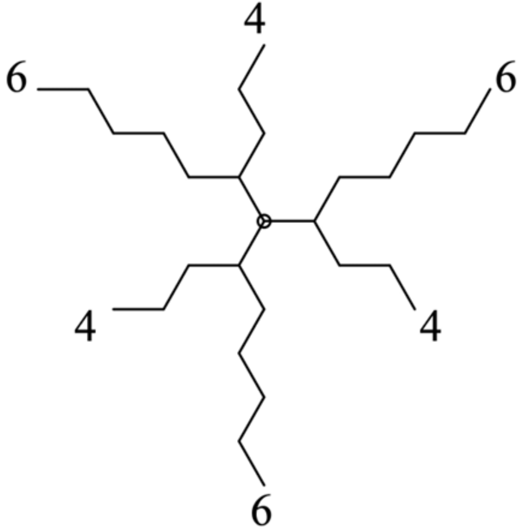
Why Benzels?

The term "Benzel" was coined in 2021 by James Propp, in honor of the chemical element benzene (whose hexagonal structure reflects the hexagonal cells of which benzels are composed), the Mercedes-Benz car company (whose logo is reminiscent of the way three hexagonal cells meet), and the inventor Gustav Benzel (whose 1870 innovation, the merry-go-round, undergoes rotation in a manner vaguely reminiscent of the three-fold rotational symmetry of benzels).

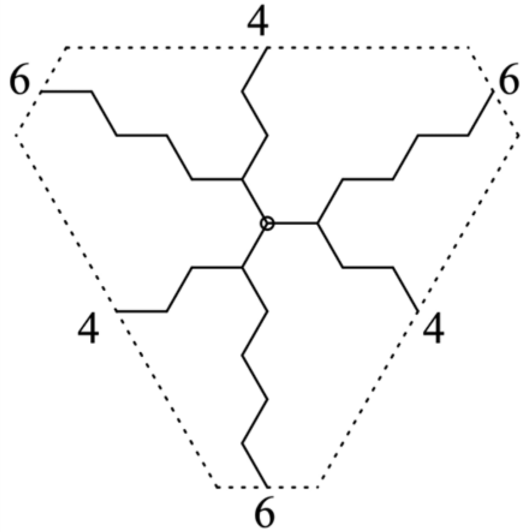
Constructing Benzels I

The question that still lingers is that how do we come up with such structures from just two given parameters. To answer the question and to go forth into any tiling of this structure thereof, we will construct a $4, 6$-benzel firsthand.

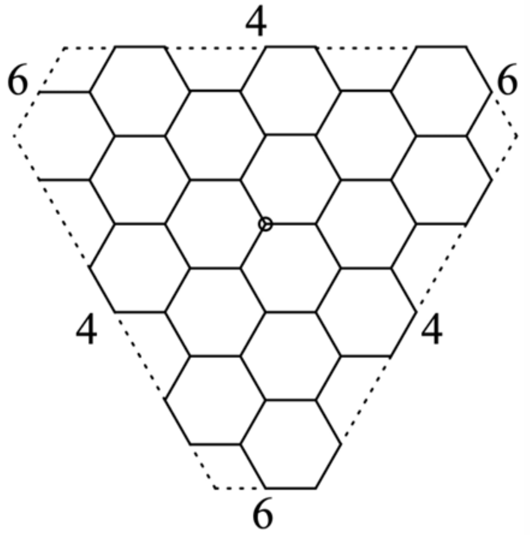
Constructing Benzels II



Constructing Benzels III



Constructing Benzels IV



Tiles in use

Being a fairly new shape, whose tilings are being considered since just around 2022, we know very less about such tilings. So we forget about the phone tiles as of now, as they have not yet been formally considered for tiling benzels.

Let's start by considering something very simple. We will start by using just one type of tiles, that is, the bones, and look at all tilings using only these.

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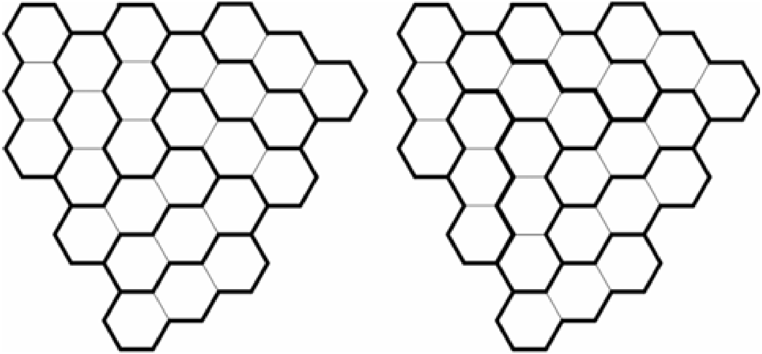
A look into tribone tiling of 5,7-benzel



The statement of the conjecture

Theorem (Propp, Kim)

The (a, b)-benzel can be tiled by tribones if and only if a and b are paired pentagonal numbers, that is,
$\{a, b\} = \{k(3k + 1)/2, k(3k - 1)/2\}$*, for some positive integer* k.

Leading to the proof

A variant of Gauss' shoelace formula allowed James Propp to compute the signed area (aka algebraic area) enclosed by a closed polygonal path and, by "twisting" the formula in the manner prescribed by the work of Conway, Lagarias, and Thurston, to compute the values of the Conway-Lagarias invariant for all benzels. This led to the proof of the conjecture.

Gauss shoelace formula

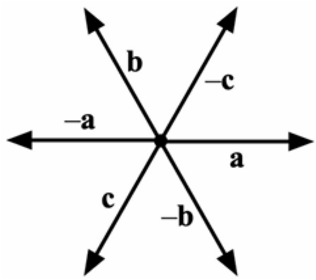
Consider a planar simple polygon with it's coordinates represented as vectors $v_1, ..., v_n$. Suppose these vectors are such that $\Sigma_{i=1}^{n} v_i = 0$. Then with $w_i = v_1 + ... + v_i$, the signed area enclosed by the vertices $w_1, ..., w_n$ is

$$
\begin{aligned}
&\frac{1}{2}\sum_i w_i \times v_{i+1} \\
=&\frac{1}{2}\sum_{1\leq i<j\leq n} v_i \times v_j \\
=&\frac{1}{2}\sum_{1\leq i<j\leq n} \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}, \text{where } v_i = (x_i, y_i)
\end{aligned}
$$

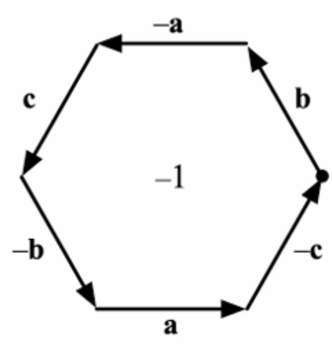
Redefining vectors for Hexagonal Grid

For the hexagonal grid we will need three unit vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. To remove factors of $\sqrt{3}$, we redefine $\times$ so that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} = +1$, $\mathbf{b} \times \mathbf{a} = \mathbf{c} \times \mathbf{b} = \mathbf{a} \times \mathbf{c} = -1$, and $\mathbf{a} \times \mathbf{a} = \mathbf{b} \times \mathbf{b} = \mathbf{c} \times \mathbf{c} = 0$. If each hexagonal cell is assigned area 1, then the signed area enclosed by a path given by the vectors $v_1, \ldots, v_n$ is $\frac{1}{6} \sum_{1 \leq i < j \leq n} v_i \times v_j$.

The New Vectors



(a) The Six Unit Vectors



(b) Path enclosing a hexagon

Navigating the Boundary of Benzels I

For a, b with $a + b \equiv c \bmod 3$, we call the (a, b)-benzel a class c benzel.

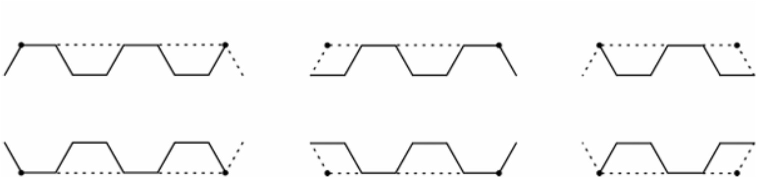


Figure 5: The upper and lower stretches of the boundary of a class 0 benzel (left), a class $+1$ benzel (middle), and a class -1 benzel (right)

Navigating the Boundary of Benzels II

Take a, b with $a + b \equiv 0 \bmod 3$. The upper stretch of the (a, b)-benzel, starting at the upper right corner of the bounding hexagon and ending at the upper left corner of the bounding hexagon is $(\mathbf{c{,}a'{,}b{,}a'})^t$ with $t = (2b - a)/3$.
The lower stretch of the (a, b)-benzel, starting at the lower left corner of the bounding hexagon and ending at the lower right corner of the bounding hexagon is $(\mathbf{a{,}c'{,}a{,}b'})^s$ with $s = (2a - b)/3$.

Navigating the Boundary of Benzels III

Thus the counterclockwise boundary of the benzel, starting and ending at the rightmost corner of the bounding hexagon, is the concatenation:

$(\mathbf{b},\mathbf{a'},\mathbf{b},\mathbf{c'})^s$ $(\mathbf{c},\mathbf{a'},\mathbf{b},\mathbf{a'})^t$ $(\mathbf{c},\mathbf{b'},\mathbf{c},\mathbf{a'})^s$ $(\mathbf{a},\mathbf{b'},\mathbf{c},\mathbf{b'})^t$ $(\mathbf{a},\mathbf{c'},\mathbf{a},\mathbf{b'})^s$ $(\mathbf{b},\mathbf{c'},\mathbf{a},\mathbf{c'})^t$

involving $12s + 12t$ unit vectors.

Just to be clear, we state that $\mathbf{a'} = -\mathbf{a}$, $\mathbf{b'} = -\mathbf{b}$, and $\mathbf{c'} = -\mathbf{c}$.

Signed area of Benzels I

The value of $\frac{1}{6}\sum_{1\leq i<j\leq n} v_i \times v_j$ can be determined in the general form and then we can fit the six undetermined coefficients. Specifically the $\binom{12s+12t}{2}$ product terms can be segregated into $\binom{6}{2}\cdot 4\cdot 4+\binom{6}{1}\cdot 4\cdot 4 = 336$ types, such that the number of terms $v_i \times v_j$ of each type is a quadratic function of s and t.

Signed area of Benzels II

This implies that the area of the benzel is itself a quadratic function of s and t, and hence a quadratic function of a and b. This quadratic function has six undetermined coefficients that can be determined by finding the area of half a dozen specific benzels and solving the resulting system of linear equations. We find in this way that the area of a class 0 benzel is equal to
$(-a^2 + 4ab - b^2 - a - b)/2$.

Signed area of Benzels III

Theorem

Let R be a benzel of class c. Then the area of R is

$$(-a^2 + 4ab - b^2 - a - b)/2 \text{ if } c = 0 \text{ or } -1$$
$$(-a^2 + 4ab - b^2 - a - b + 2)/2 \text{ if } c = +1$$

Shadow Paths I

Conway and Lagarias show that in any simply-connected region R that can be tiled by stones and bones, the number of right-pointing stones minus the number of left-pointing stones depends only on the region being tiled, not the particular tiling. We call this the *rescaled Conway-Thurston invariant* and denote it by $i(R)$.

Shadow Paths II

We start at some point p on the boundary of R. Let $e_0, ..., e_n$ be a counterclockwise path from p and back to itself traversing the boundary of R. Let us call the path π. For $1 \leq i \leq n$, π weaves at step i if e_{i-1} and e_{i+1} are parallel, and π winds at step i if e_{i-1}, e_i, e_{i+1} are consecutive edges of a hexagon.

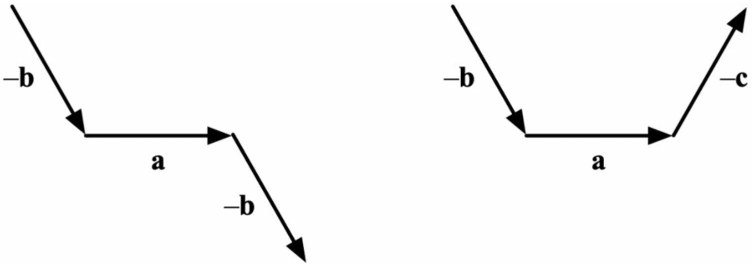
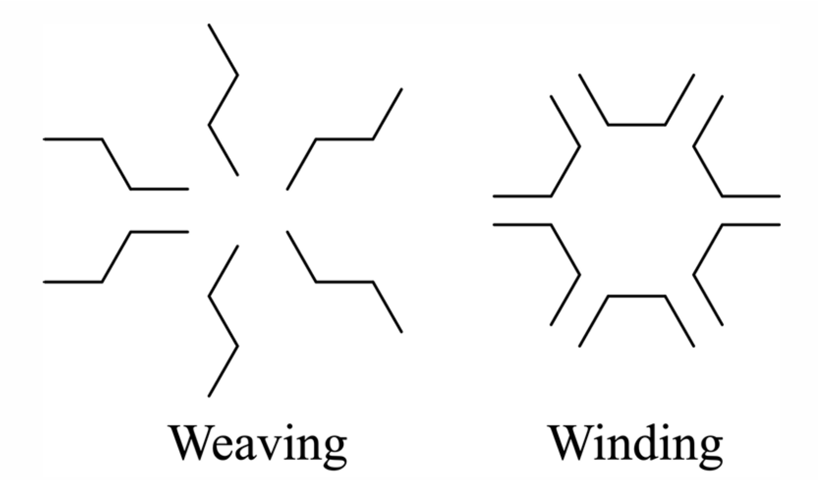


Figure 6: Weaving and Winding

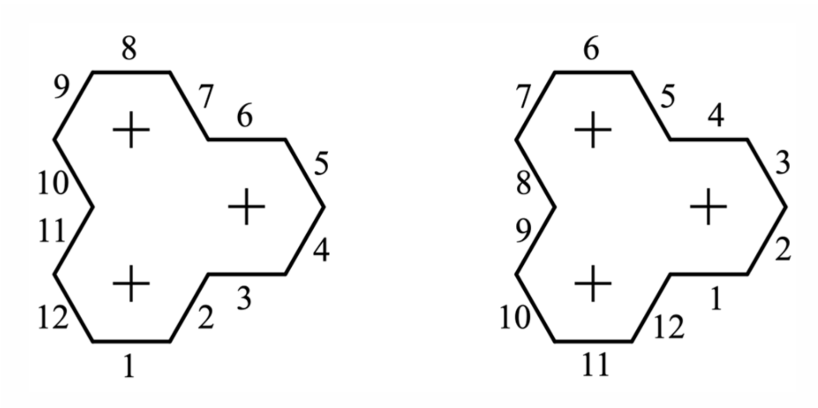
Shadow Paths III

Now we determine the shadow path π'. The shadow path π' winds where π weaves, and π' weaves where π winds.



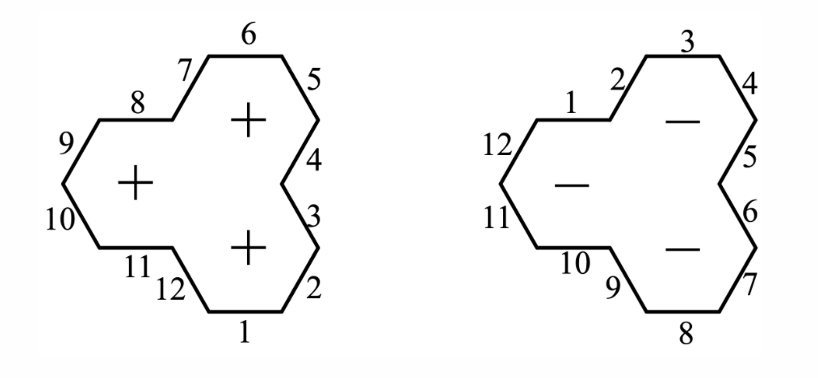
Conway-Lagaris Invariant of Various Tiles I

Thus, the Conway-Lagaris Invariant of a right stone is $+3$.



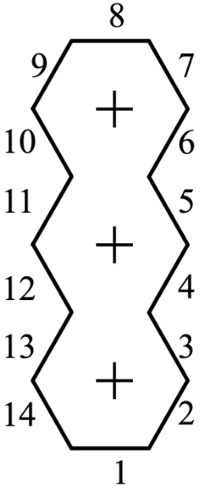
Conway-Lagaris Invariant of Various Tiles II

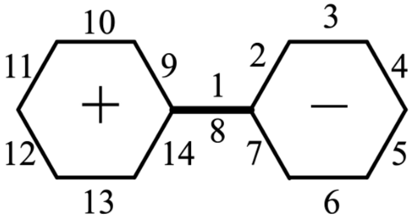
Thus, the Conway-Lagaris Invariant of a left stone is -3.



Conway-Lagaris Invariant of Various Tiles III

Thus, the Conway-Lagaris Invariant of a tribone is 0.





Conway-Lagarias Invariant of different classes of Benzels

Theorem

Let R be a benzel of class c. Then $I(R)$, the (unrescaled) Conway Lagarias invariant of R, is

$$(-3a^2 + 6ab - 3b^2 + a + b)/2 \text{ if } c = 0$$
$$(a^2 - 4ab + b^2 + a + b - 2)/2 \text{ if } c = +1$$
$$(-3a^2 + 6ab - 3b^2 - a - b + 2)/2 \text{ if } c = -1$$

This is fairly easy to check and fairly algebraic and so, can be proven easily. (Try for yourself)

Answering the conjecture

To tile a benzel by just using tribone tiles, the Conway-Lagaris invariant of the benzel must be 0.
Now, it is fairly simple and number-theoretic in nature to check that $c = 0$ is the only case where the Conway-Lagaris invariant of the benzel can be 0, and we can deduce that $\{a, b\} = \{k(3k+1)/2, k(3k-1)/2\}$, for some positive integer k.

References

1. A Pentagonal Number Theorem for Tribone Tilings - James Propp, Jesse Kim
2. Tilings with Polyominoes and Combinatorial Group Theory - J.H. Conway, J.C. Lagarias
3. Conway's Tiling Groups - William P. Thurston
4. Tilings - Frederico Ardila, Richard P. Stanley